EXPONENTIAL DECAY OF MATRIX COEFFICIENTS

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1. Introduction

We will work in the setting $G = \mathrm{PSL}(2,\mathbb{R})$ and $\Gamma < G$ a discrete subgroup, not virtually cyclic. Define $a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$. We have a Cartan decomposition for $K = \mathrm{PSO}(2)$, $G = KA^+K$. For $f_1, f_2 \in C_c(\Gamma\backslash G)$ we consider $t \mapsto \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\,\mathrm{d}x$. We want to answer the same 3 questions in the setting where Γ is not a lattice.

We will operate under the following assumptions:

(1) $\mathrm{vol}(\Gamma\backslash G) = \infty$
(2) Γ is finitely generated, this is the same as saying $\Gamma\backslash\mathbb{H}$ is of finite type

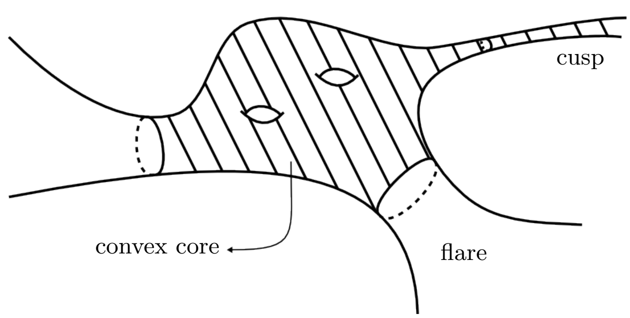

Figure 1. $\Gamma\backslash\mathbb{H}$

Cusps may or may not exist, but there will always be flares because of the volume assumption.

There is a natural correspondence between $T^1(\Gamma\backslash\mathbb{H})$ and $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ where geodesic flow for time t precisely corresponds to right translation by a_t.

We define the following invariants:

Definition 1.

(1) Λ_Γ = the *limit set* of Γ = the set of all accumulation points of $\Gamma(0) \subset \mathbb{S}^1 = \partial_\infty(\mathbb{H}^2)$.
(2) δ_Γ = the *critical exponent* of Γ = Hausdorff dimension of Λ_Γ (Theorem of Patterson-Sullivan '76).
(3) ν_0 = *Patterson-Sullivan measure* on Λ_Γ = δ-dimensional measure on Λ_Γ (if there are no cusps).

Under our assumptions we have that $0 < \delta_\Gamma < 1$, since Γ is not elementary and not a lattice.

Definition 2 (Bowen-Margulis-Sullivan measure)**.**

$$\mathrm{d}\widetilde{m}^{\mathrm{BMS}}(g) = \frac{\mathrm{d}\nu_0(g^+)\,\mathrm{d}\nu_0(g^-)\,\mathrm{d}t}{|g^+ - g^-|^{2\delta}}$$

This measure is left Γ-invariant and right A-invariant, and induces a measure, m^{BMS}, on $\Gamma\backslash G$.

Theorem 1 (Sullivan)**.** $m^{BMS}(\Gamma\backslash G) < \infty$.

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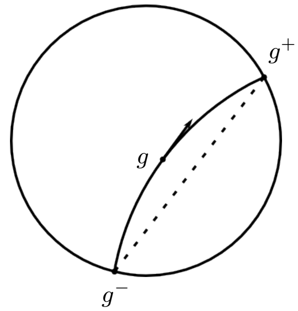

FIGURE 2. The unique geodesic determined by the vector g

This theorem essentially says that the Bowen-Margulis-Sullivan measure is supported on the convex core. Sullivan also proved that this measure is ergodic.

2. Limit of the correlation function

Theorem 2 (Rudolph '82, Babillot '02). *For $f_1, f_2 \in C_c(\Gamma\backslash G)$,*

$$\int f_1(xa_t)f_2(x)\,dm^{BMS}(x) \xrightarrow[t\to\infty]{} \frac{1}{m^{BMS}(\Gamma\backslash G)} \cdot m^{BMS}(f_1)m^{BMS}(f_2)$$

We define the Burger-Roblin measure, m^{BR}, the unique nontrivial ergodic measure for the action of the expanding horocyclic subgroup and the measure m^{BR_*}, the unique nontrivial ergodic measure for the action of the contracting horocyclic subgroup. Then we have the following theorem:

Theorem 3 (Roblin '03).

$$e^{(1-\delta)t}\int f_1(xa_t)f_2(x)\,dx \xrightarrow[t\to+\infty]{} \frac{1}{m^{BMS}(\Gamma\backslash G)} \cdot m^{BR}(f_1)m^{BR_*}(f_2)$$

3. Exponential rate

There are two cases we need to consider, when $\delta > \frac{1}{2}$ and when $\delta \leq \frac{1}{2}$. Let $\Delta =$ (negative) Laplacian on $L^2(\Gamma\backslash\mathbb{H})$ so that its spectrum $\sigma(\Delta) \subset [0,\infty)$.

Theorem 4 (Patterson, Lax-Phillips '82).

(1) *If $\delta > \frac{1}{2}$, $\sigma(\Delta) = \sigma_p(\Delta) \cup \sigma_c(\Delta)$, a union of a descrete point spectrum $\sigma_p(\Delta) = \{0 < \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_m < \frac{1}{4}\}$ and a purely continuous spectrum $\sigma_c(\Delta) = \left[\frac{1}{4},\infty\right)$. Patterson showed that $\lambda_0 = \delta(1-\delta)$.*
(2) *If $\delta \leq \frac{1}{2}$, $\sigma(\Delta) = \sigma_c(\Delta) = \left[\frac{1}{4},\infty\right)$.*

Definition 3. The *spectral gap* is defined to be $\lambda_1 - \lambda_0$.

3.1. Case 1: $\delta > \frac{1}{2}$.

To formulate the exponential error term statement, it will be convenient to reparametrize the eigenvalues. Write $\lambda_1 = S_1(\Gamma)(1 - S_1(\Gamma))$, where we choose a unique $\frac{1}{2} < S_i < \delta$.

In this case, Theorem 4 is equivalent to the statement that $L^2(\Gamma\backslash G) = \mathcal{H}_{S_0} \oplus \mathcal{H}_{S_1} \oplus \cdots \oplus \mathcal{H}_{S_m} \oplus$ "tempered". We can think of $\mathcal{H}_{S_0}$ as the "minimal" representation.

Theorem 5 (Bourgain-Kontorovich-Sarnak, Mohammadi-Oh). *If $\delta > \frac{1}{2}$, then, for $f_1, f_2 \in C_c^\infty(\Gamma\backslash G)$,*

$$e^{(1-\delta)t}\int f_1(xa_t)f_2(x)\,dx = \frac{1}{m^{BMS}(\Gamma\backslash G)} \cdot m^{BR}(f_1)m^{BR_*}(f_2) + O(S_1(f_1)S_1(f_2)e^{-(1-\varepsilon)(\delta - S_1(\Gamma))t}$$

Let $\Gamma < \mathrm{SL}(2,\mathbb{X})$ be a finitely generated, noncyclic subgroup. For q we define

$$\Gamma(q) = \{\gamma \in \Gamma \mid \gamma \equiv 0 \mathrm{mod}\ q\}$$

Fix a finite symmetric generating set S of Γ with $|S| = k$. Then we define the *Cayley graph* $\mathcal{G}(\Gamma(q)\backslash\gamma, S)$ to be the graph whose vertices are elements in $\Gamma(q)\backslash\Gamma = \{v_1, v_2, \ldots, v_{m_q}\}$ and where edges are elements of the set $\{(\gamma, \gamma s) \mid \gamma \in \Gamma(q)\backslash\Gamma, s \in S\}$. We have an adjacency matrix

$$A_q = (a_{ij}) = \begin{cases} 1 & \text{if } \{v_1, v_2\} \in \text{ Edge} \\ 0 & \text{otherwise} \end{cases}$$

and a *combinatorial Laplacian* $\widetilde{\Delta}_q = K \setminus A_q$, it has eigenvalues $0 < \widetilde{\lambda}_1(q) \leq \widetilde{\lambda}_2(q) \leq \cdots$. If $\inf_q \widetilde{\lambda}_1(q) > 0$, we have an *expander family.*

Theorem 6 (Bourgain-Gamburd for prime moduli '06, Bourgain-Gamburd-Sarnak for square-free moduli '11)**.** *For $\Gamma < SL(2, \mathbb{X})$ be a finitely generated, noncyclic subgroup,*

$$\{\mathcal{C}(\Gamma(q)\backslash\Gamma, S) \mid q \textit{ is square free}\}$$

forms an expander *family.*

Theorem 7 (Transfer principle, Bourgain-Gamburd-Sarnak)**.** *If $\delta > \frac{1}{2}$, the combinatorial spectral gap is the same as the archimedian spectral gap,*

$$\inf_q \widetilde{\lambda}_1(q) > 0 \iff \inf_q \lambda_1(q) > \delta(1-\delta)$$

3.2. **Case 2:** $\delta \leq \frac{1}{2}$**.** From now on we assume Γ is convex cocompact (i.e. no cusps).

Theorem 8 (Dolgopyat '98, Stoyanov 2011)**.**

$$\int_{\Gamma\backslash G} f_1(xa_t)f_2(x)\, dm^{BMS}(x) = \frac{1}{|m^{BMS}|} m^{BMS}(f_1)m^{BMS}(f_2) + O(\|f_1\|_{C^1}\|f_2\|_{C^1}e^{-\eta t})$$

So we have exponential mixing of the BMS measure.

Theorem 9 (Oh-Winter)**.**

$$e^{(1-\delta)t}\int_{\Gamma\backslash G} f_1(xa_t)f_2(x)\, dx = \frac{1}{|m^{BMS}|} m^{BR}(f_1)m^{BR_*}(f_1) + O(\|f_1\|_{C^1}\|f_2\|_{C^1}e^{-\eta t})$$

here the error term depends on the support of f_1 and f_2.

Theorem 10 (Oh-Winter)**.** *Let $\Gamma < SL(2, \mathbb{Z})$, convex cocompact, then there exists $\eta > 0$ and $c \geq 3$ such that for all square-free q (with no small divisors) we have*

$$e^{(1-\delta)t}\int_{\Gamma(q)\backslash G} f_1(xa_t)f_2(x)\, dx = \frac{1}{|m^{BMS}|} m^{BR}(f_1)m^{BR_*}(f_1) + O(q^c\|f_1\|_{C^1}\|f_2\|_{C^1}e^{-\eta t})$$

We define a resolvant operator

$$R_q(s) := (\Delta - s(1-s))^{-1} : C_c^\infty(\Gamma(q)\backslash\mathbb{H}) \to C_c^\infty(\Gamma(q)\backslash\mathbb{H})$$

It follows from the work of Patterson that $R_q(s)$ is holomorphic for $\Re(s) > \delta$ and $s = \delta$ is the unique pole on $\Re(s) = \delta$ of rank 1. The resonances are then the poles of the meromorphic continuation of the resolvant. For each $q \in \mathbb{N}$, Naud showed that there exists $\varepsilon_q > 0$ such that $\{\Re(s) > \delta - \varepsilon_q\}$ is resonance-free, except for $s = \delta$.

Theorem 11 (Oh-Winter)**.** *There exists $\varepsilon > 0$ such that for all q square-free, $\Re(s) > \delta - \varepsilon\}$ is resonance-free for $R_q(s)$.*