UNIPOTENT FLOWS ON INFINITE VOLUME MANIFOLDS

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Joint work with H. Oh and C. McMullen.

1. Introduction

What from the existing theory are we able to generalize? It turns out that

(*) Horospherical case - most things work
(**) Non horospherical case - almost nothing works

1.1. **Setup.** Let $M = \mathbb{H}^3/\Gamma$, where Γ is torsion free and convex cocompact.

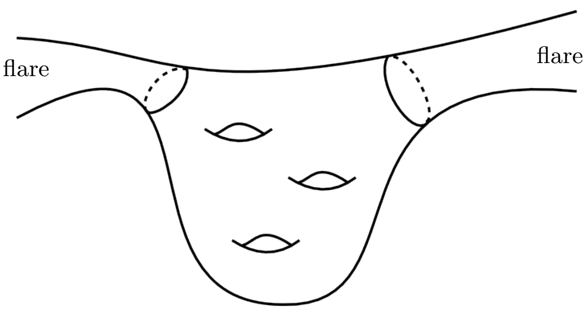

Figure 1. The surface M

Question 1. Given a function $f : \mathbb{H}^2 \to M$, how can we describe $\overline{f(\mathbb{H}^2)}$?

1.2. **Previous results.** Work of Ratner, Margulis, Dani-Margulis, Shah, answer this and a much more general question

Theorem 1. *If M is compact, then $f(\mathbb{H}^2)$ is either closed or dense*

2. Generalization to non-compact M

How much of this can be generalized to the case when M is not compact? Not much.

Example 1 (Why this doesn't hold in general)**.** Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a cocompact, torsion free lattice, we can think of it as sitting in $PSL(2,\mathbb{C})$, and we get a surface $\mathbb{H}^2/\Gamma = X$. Here $M = \mathbb{H}^2\Gamma$ and is diffeomorphic to $X \times \mathbb{R}$. Let $f : \mathbb{H}^2 \to M$ then $f(\mathbb{H}^2) \subset M$, $f(\mathbb{H}^2) = \gamma \times \mathbb{R}$. Suppose $\widetilde{\gamma}$ is such that $\bar{\gamma}$ is a fractal, then $\overline{f(\mathbb{H}^2)} = \bar{\gamma} \times \mathbb{R}$. Therefore, in general the picture is not as nice.

Let $\mathcal{M} = \mathbb{H}^3/\Gamma$, then $T^1\mathcal{M} = M\backslash\mathrm{PSL}(2,\mathbb{C})/\Gamma$ where $\Gamma \subset \mathrm{PSL}(2,\mathbb{C}) = G$, the frame bundle $F\mathcal{M} = \mathrm{PSL}(2,\mathbb{C})/\Gamma$. We define the following subgroups

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}, N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$$

and $H = \mathrm{PSL}(2,\mathbb{R}) \subset G$.

Theorem 1 is really asking about H orbits,

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Theorem 2. *Every H orbit in G/Γ is closed or dense, if Γ is a lattice.*

The proof uses the dynamics of the U-action on G/Γ.

Theorem 3 (Ratner)**.** *if μ is a U-invariant probability measure on G/Γ then μ is nice.*

Proof. Two main ingredients:

(1) Algebraic ingredient: Let (ρ, v) be any finite dimensional representation of G, and $v \in V$, then the map $t \mapsto \rho(u_t)v$ is a polynomial in t. Consider the points x and gx, we flow by u_t and consider the displacement, $u_t g u_t^{-1} u_t x = u_t g x$

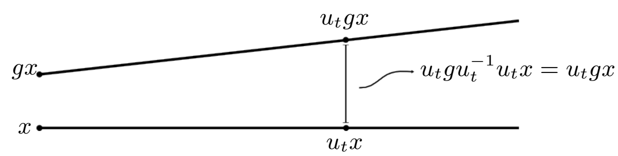

FIGURE 2. The u_t flow of x and g^x

If we let $g = \exp(r)$ where $r \in \mathfrak{sl}(2, \mathbb{C})$, then the displacement is really $\exp(\mathrm{Ad}(u_t)r)u_t x = i_t g^x$. The map $\mathrm{Ad} : U \to \mathfrak{g}$, $t \mapsto \mathrm{Ad}(u_t)r$, governs the displacement. $\mathrm{Ad}(u_t)r$ is a polynomial which means that it grows very slowly.

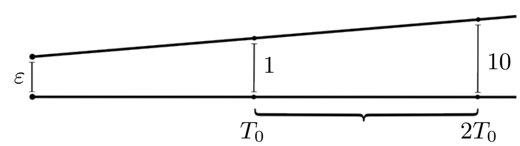

FIGURE 3. Slow growth of displacement

So we would like the pieces of the u_t orbits between T_0 and $2T_0$ to mimic the behavior of the entire orbit. Then we'd have two pieces of the orbit that are nearly parallel to each other, and the translation between them should leave the object of study invariant.

(2) **Birkhoff ergodic theorem**: For $f \in C_c(G/\Gamma)$,

$$\frac{1}{T}\int_0^T f(u_t x)\,\mathrm{d}t \to \int f\,\mathrm{d}\mu$$

Since the normalizing factor is linear, we could also consider $\int_{T/2}^{T}$ and normalize appropriately by $\frac{2}{T}$, thereby restricting to a small window.

If we do not know that the orbits in the window return to a compact set, then the fact that the flow diverges slowly, indicates we are going into the flare, and this raises an issue. We can try to use the following,

Hopf ratio theorem: Take $f, g \in C_c(G/\Gamma)$, μ is Radon, U ergodic and invariant, then

$$\frac{\int_{-T}^{T} f(u_t x)\,\mathrm{d}t}{\int_{-T}^{T} g(u_t x)\,\mathrm{d}t} \to \frac{\mu(f)}{\mu(g)}$$

□

Now the question is whether or not we can upgrade this to a window. Unfortunately it seems like this is not a general phenomenon.

Question 2. Is it true in general that if (X, μ) is U-invariant ergodic infinite measure that,

$$\frac{\int_{[T,-T]\setminus[rT,-rT]} f(u_t x)\,\mathrm{d}t}{\int_{T,-T]\setminus[rT,-rT]} g(u_t x)\,\mathrm{d}t} \to \frac{\mu(f)}{\mu(g)}$$

Most likely the answer is "no". You can impose condition on Γ to make this hold.

2.1. **Some results.**

(1) Flaminio-Spatzier: Γ_1, Γ_2 are convex cocompact (measurable with respect to the geometric measure). $\varphi : T^1(\mathbb{H}^3/\Gamma_1) \to T^1(\mathbb{H}^3/\Gamma_2)$ such that
- φ takes unstable to unstable
- Isometry on each unstable horocycle

Then φ "comes" from an isometry

(2) Burger: $\mathrm{PSL}(2,\mathbb{R})$, $\delta > \frac{1}{2}$; Roblin: general case - The action of the full horospherical subgroup is rigid and there is exactly one new measure in T^1. (Γ is always Zariski dense)

3. Examples where there is rigidity for $\overline{f(\mathbb{H}^2)}$.

Let M be a cylindrical convex cocompact with geodesic boundary and Γ such that $\Lambda(\Gamma) = \overline{\Gamma \cdot o} \subset \partial\mathbb{H}^3$.

(*) $\partial\mathbb{H}^3 \setminus \Lambda(\Gamma) = \bigsqcup D_i$ Where $\{D_i\}$ is a collection of infinity many disjoint round disks.

(**) $\mathrm{hull}(\Gamma)$ = convex hull of $\Lambda(\Gamma)$. $\mathrm{core}(\Gamma) = \mathrm{hull}(\Gamma)/\Gamma$. We want $\partial\mathrm{core}(\Gamma)$ = geodesic surface.

Theorem 4. *M is cylindrical convex cocompact with geodesic boundary, $f : \mathbb{H}^2 \to M$ then $\overline{f(\mathbb{H}^2)}$ is one of the following*

(I) $\overline{f(\mathbb{H}^2)} = f(\mathbb{H}^2)$ *is a closed surface*

(II) $\overline{f(\mathbb{H}^2)} = f(\mathbb{H}^2)$ *is a proper embedding of a surface with infinite area*

(III) $f(\mathbb{H}^2)$ *is dense, i.e.* $\overline{f(\mathbb{H}^2)} = M$.

(IV) $\overline{f(\mathbb{H}^2)} = f(\mathbb{H}^2)$ *is an embedding of* $\mathbb{H}^2$ *in an end*

(V) $\overline{f(\mathbb{H}^2)}$ *is one of the ends*

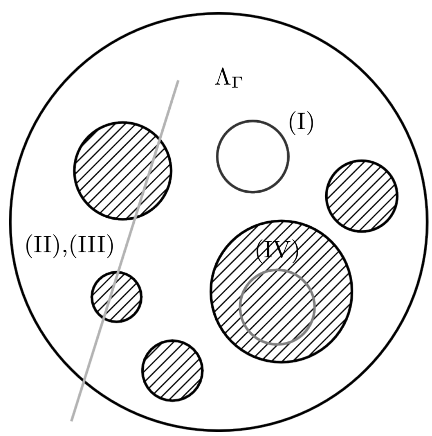

Figure 4. The limit set Λ_Γ and $\overline{f(\mathbb{H}^2)}$