UNIPOTENT FLOWS ON INFINITE VOLUME MANIFOLDS

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1. Introduction

Recall the setting from yesterday: $G = \mathrm{PSL}(2,\mathbb{C})$, and Γ is convex, cocompact, Zariski dense, torsion free, and not a lattice. Denote the *limit set* by $\Lambda(\Gamma)$, $\dim \Lambda(\Gamma) = \delta < 2$, where δ is the *critical exponent*. We have the following subgroups

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}, N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}$$

The tangent space, $T^1(\mathbb{H}^3)$ can be identified to $(\partial\mathbb{H}^3 \times \partial\mathbb{H}^3 \setminus \Delta\partial\mathbb{H}^3) \times \mathbb{R}$ by $x \mapsto (x^-, x^+, t)$. We would like to consider the N action on $\mathrm{PSL}(2,\mathbb{C})/\Gamma$ and describe the N-invariant ergodic Radon measures.

Fact 1. *If Γ is not a lattice, then the Haar measure is never ergodic.*

Proof. If $\mathcal{H}(x)$ is a horosphere based at $x^- \notin \Lambda(\Gamma)$ then $\mathcal{H}(x) \to T^1(\mathbb{H}^3/\Gamma)$ escapes to infinity. Since $\delta > 2$, Haar almost every N orbit never returns, and hence is never ergodic. ☐

Now the question becomes, is there any interesting measure that is actually ergodic? The answer is yes, there is a very natural geometrically constructed measure that is invariant and ergodic for the N action.

2. The BR measure

The BR measure is supported on the set $\{x \mid x^- \in \Lambda(\Gamma)\}$ and is N and Γ invariant. Take Patterson Sullivan measure on $\Lambda(\Gamma)$ (δ-dimensional Hausdorff measure), it is a family of Γ-conformal densities $\{\nu_x \mid x \in \mathbb{H}^3\}$ supported on $\partial\mathbb{H}^3$. This means that

$$\frac{\mathrm{d}\nu_{\gamma x}}{\mathrm{d}\nu_x}(\xi) = e^{-\delta\beta_\xi(\gamma x, x)}$$

for $\xi \in \partial\mathbb{H}^3$, $\gamma(t)$ a geodesic from 0 to ξ and $\beta_\xi(y,z) = \lim_{t\to\infty} d(y,\gamma(t)) - d(z,\gamma(t))$.

Similarly we have $\{m_x \mid x \in \mathbb{H}^3\}$, a rotation invariant measure. So we can try to use these two measures to induce a measure on $T^1(\mathbb{H}^3/\Gamma)$. We define the following:

$$\mathrm{d}m^{\mathrm{BR}}(x) = e^{2\beta_{x^+}(0,\pi x)} e^{\delta\beta_{x^-}(0,\pi x)}\, \mathrm{d}\nu_0(x^-)\, \mathrm{d}m_0(x^+)\, \mathrm{d}t$$
$$\mathrm{d}m^{\mathrm{BMS}}(x) = e^{2\beta_{x^+}(0,\pi x)} e^{\delta\beta_{x^-}(0,\pi x)}\, \mathrm{d}\nu_0(x^-)\, \mathrm{d}\nu_0(x^+)\, \mathrm{d}t$$
$$\mathrm{d}m^{\mathrm{Haar}}(x) = e^{2\beta_{x^+}(0,\pi x)} e^{\delta\beta_{x^-}(0,\pi x)}\, \mathrm{d}m_0(x^-)\, \mathrm{d}m_0(x^+)$$

This measure is N-invariant, but is not invariant under the geodesic flow. So $a_t * m^{\mathrm{BR}} = e^{(\delta-2)t} m^{\mathrm{BR}}$ on $T^1(\mathbb{H}^3/\Gamma)$. Thus this is an infinite measure whenever $\delta < 2$, in fact m^{BR} is finite if and only if Γ is a lattice.

Theorem 1 (Burger $n = 2$, Roblin in general)**.** *m^{BR} is the only new measure.*

Even though m^{BR} is an infinite measure, the measure that governs the dynamics is actually m^{BMS}, which is a finite probability measure.

Theorem 2 (Mixing Theorem)**.** *Let $\psi_1, \psi_2 \in C_c(G/\Gamma)$, then*

$$e^{(2-\delta)t} \int \psi_1(a_t x)\psi_2(x)\, dm^{BR}(x) \to m^{BR}(\psi_1) m^{BMS}(\psi_2)$$

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Theorem 3 (Equidistribution Theorem)**.** *Let* $\psi \in C_c(G/\Gamma)$*,* $x^- \in \Lambda(\Gamma)$*, then*

$$\frac{1}{\mu_x^{PS}(B(T))} \int_{B(T)} \psi(n_z x)\, dz \to m^{BR}(\psi)$$

and

$$d\mu_{N_g}^{PS} = e^{\delta \beta_{(u_t g)^+}(0, \pi(u_t g))}\, d\nu_0\left((u_t g)^+\right)$$

Proof. Let $t = \log T$,

$$\begin{aligned} \frac{1}{\mu_x^{\mathrm{PS}}(B(T))} \int_{B_N(T)} \psi(n_z x)\,\mathrm{d}z &= \frac{1}{\mu_x^{\mathrm{PS}}(B(T))} \int_{B_N(1))} \psi(a_t n_z a_{-t} x) e^{2t}\,\mathrm{d}z \\ &= \frac{e^{2t}}{e^{\delta t} \mu_{a_{-t}x}^{\mathrm{PS}}(B(1))} \int_{B_N(1)} \psi(a_t n_z(a_{-t}x))\,\mathrm{d}z \\ &= \frac{e^{(2-\delta)t}}{\mu_{a_{-t}x}^{\mathrm{PS}}(B_N(1))} \int_{B_N(1)} \psi(a_t n_z(a_{-t}x))\,\mathrm{d}z \end{aligned}$$

We need to consider what happens to $\partial B_N(1)$, taking what we've found here and doing some work will imply equidistribution. □

Return times of $\{u_z x \mid z \in B_U(T)\}$ are comparable to

$$c^{-1} T^\delta < \mu_x^{\mathrm{PS}}(B_N(T)) < c T^\delta$$

for some $c > 1$.

So the "window like" theorem holds for N orbits.

Let $\Omega = \{g \mid g^\pm \in \Lambda(\Gamma)\}$. Then the return of U orbits to Ω is the same as asking for $\{(u_t x)^+ \in \Lambda(\Gamma)\}$

Consider the U action with respect to BR measure, m^{BR} is ergodic for U if and only if $\delta > 1$. (m^{BR} is ergodic for N action on G/Γ: Winter).

Question 1. Are there any other interesting U ergodic measures with Zariski dense support?

If Γ is convex cocompact, there exists and infinite countable collection of round (open) disks $\{B_i\}$ and $\Lambda(\Gamma) = S^2 \setminus \sqcup_{i=1}^\infty B_i$

Lemma 1. *There exists* k*,* T_x *such that* $\{u_t \mid t \in [-T, T] \setminus [-kT, kT]\} \cap \Omega \neq \emptyset$ *for* $T > T_x$*.*

Proof. Let $g^- \in \Lambda(\Gamma) \setminus \sqcup \partial B_i$, $L > 1$ we want to show that

$$B(g^-, Lr) \setminus B(x^-, r) \cap \Lambda(\Gamma) \neq \emptyset$$

if not, $B(g^-, Lr) \setminus B(g^-, r) \in \sqcup B_i$. Claim: This is not possible. There exists a unique i such that $B(g^-, Lr) \setminus B(g^-, r) \in B_i$. Suppose not, then $\mathrm{dist}(\mathrm{hull}(D_1), \mathrm{hull}(D_2)) \to 0$, and this is a contradiction. □