MARTIN BOUNDARY AND LOCAL LIMIT THEOREM OF BROWNIAN MOTION ON NEGATIVELY CURVED MANIFOLDS

SEONHEE LIM

Joint work with F. Ledrappier

1. The heat kernel

Consider X, a Riemannian manifold, open, connected and complete.

Definition 1. We call the function $p(t,x,y)$ the *heat kernel* if it satisfies the following properties:

- it is the fundamental solution to the heat equation, i.e.

$$\frac{\partial}{\partial t} p(t,x,y) = \Delta_x p(t,x,y)$$

- and, given an initial condition,

$$u(x,t) = \int p(t,x,y) u(y)\, \mathrm{d}y \xrightarrow{t\to 0} u(x)$$

Let $-\lambda_0$ be the maximal eigenvalue in $L^2(X)$, where λ_0 is the bottom of $\mathrm{Sp}(-\Delta)$. Assume for the moment that we have discrete spectrum, that is we can write $p(t,x,y,) = \sum e^{-\lambda_i t}\phi_i(x)\phi_i(y)$, where ϕ_i are the $-\lambda_i$ eigenfunctions of the Laplacian. In this case, we do not have discrete spectrum, but we are still able to view, by the spectral theorem,

$$-\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log p(t,x,y)$$

which says that $-\lambda_0$ is the exponential growth rate of the heat kernel.

Theorem 1 (Ledrappier-Lim)**.** *If $X = \widetilde{M}$, M a compact manifold CAT(-1), then as $t \to \infty$.*

$$p(t,x,y) \sim e^{-\lambda_0 t} t^{-3/2} C(x,y)$$

where $C(x,y) > 0$. More precisely,

$$\lim_{t\to\infty} e^{\lambda_0 t} t^{3/2} p(t,x,y) = C(x,y)$$

Examples.

(1) $\mathbb{R}^d$:

$$p(t,x,y) = c t^{-d/2} e^{\frac{-(d(x,y))^2}{4t}} \sim t^{-d} 2$$

(2) $\mathbb{H}^3$:

$$p(t,x,y) \sim e^{-t} t^{-3/2} \frac{d(x,y)}{\sinh d(x,y)}$$

(3) (Bougerol, 81) G/K a symmetric space:

$$p(t,x,y) \sim e^{-\lambda_0 t} t^{-\frac{\mathrm{rk} G + \# \mathrm{roots}}{2}} \Phi(x,y)$$

where $\Phi(x,y)$ is the Harish-Chandra function (K_x-invariant).

We have more information than just Theorem 1,

Theorem 2 (Anker-Bougerol-Jeulin, 2002)**.** *If $\lim_{t\to\infty} \dfrac{p(t,x,y)}{p(t,x,x)}$ exists, then $C(x,y)$ is a $(-\lambda_0)$-eigenfunction.*

Date: February 06, 2015.

Conjecture 1 (Davies, '97)**.** $\lim_{t\to\infty} \frac{p(t,x,y)}{p(t,x,x)}$ *always exists.*

Corollary 1 (Ledrappier-Lim)**.** *Davies' conjecture holds for* $\widetilde{M}$*.*

Proof of asymptotic in Thm 1. It is enough to show

$$\int_0^\infty e^{-st} t e^{\lambda_0 t} p(t,x,y)\,\mathrm{d}t \sim s^{-1/2} C(x,y),$$

since by using the Tauberian theorem the statement is equivalent to showing

$$\int_0^T t^{\lambda_0 t} p(t,x,y)\,\mathrm{d}t \sim T^{-1/2} C(x,y).$$

It is an exercise to prove that the statement we want, $e^{-\lambda_0 t} p(t,x,y) \sim t^{-3/2} C(x,y)$ then follows. Define the λ-Green function, $G_\lambda(x,y) = \int_0^\infty e^{\lambda t} p(t,x,y)\,\mathrm{d}t$, then

$$\int_0^\infty e^{-st} t e^{\lambda_0 t} p(t,x,y) = \frac{\partial}{\partial \lambda} G_\lambda(x,y)\bigg|_{\lambda=\lambda_0 - s}.$$

We want to understand the behavior of this derivative. □

2. Counting geodesics

[Margulis, Ledrappier, Hamenstadt, Roblin, Parry-Pollicot, Paulin-Pollicot-Schapira]

Let M be a compact, negatively curved manifold. Fix two points x, y, we would like to count

$$\#\left\{\text{geodesics } \widehat{xy} \text{ of length } \in [t, t+\delta]\right\} = \sum_{\substack{\gamma\in\Gamma \\ t\le d(x,y)\le t+\delta}} 1 = \sum_{\substack{A\subset T_x^1 M \\ B\subset T_y^1 M}} \#\left(B \cap \bigcup_{t\le s\le t+\delta} g_s A\right)$$

Thicken A, B to $\tilde{A}, \tilde{B}$ so that $\#\left(B\cap \bigcup_{t\le s\le t+\delta} g_s A\right) = \#\left(\tilde{B}\cap g_t \tilde{A}\right)$. Then

$$\mu(\tilde{B}\cap g_t\tilde{A}) = u_B e^{-ht} s_A \delta_A \#(\tilde{B}\cap g_t \tilde{A})$$

Since geodesic flow is mixing with respect to μ,

$$\mu(\tilde{B}\cap g_t\tilde{A}) \to \mu(\tilde{B})\mu(\tilde{A}) = u_B u_A s_B s_A \delta_B \delta_A.$$

Thus $\#(\tilde{B}\cap g_t\tilde{A}) \to e^{ht}\delta u_A s_B$ and so

$$\sum_{\substack{\gamma\in\Gamma \\ t\le d(x,y)\le t+\delta}} 1 \to \delta e^{ht} \|\mu_x\| \|\mu_y\|$$

Remark 1. $\mu = m^{\mathrm{BMS}}$ and attains $\sup_{\mu, g^t-\mathrm{inv}} \{h_\mu\}$

2.1. Counting geodesics with weights.

Let $F : T^1 M \to \mathbb{R}$, a Holder continuous function, be the *potential*. Our sum is now $\sum_{\gamma\in\Gamma} e^{\int_x^y F}$. We divide into pieces so that the weight, $e^{\int_x^y F}$ is almost constant on each A and B. Then thicken to $\tilde{A}, \tilde{B}$ and we get the same asymptotic, with different rate of growth, ht becomes $P(F)t$.

Remark 2. If $F \neq 0$, $\mu = \mu_F$ attains $\sup_{\mu, g^t-\mathrm{inv}} \left\{h_\mu + \int F\,\mathrm{d}\mu\right\}$

3. Understanding the λ-Green function

For each $\lambda < \lambda_0$, choose F_λ so that

$$e^{\int_x^y \tilde{F}} = k^2 \lambda(x,y,\xi) = \left(\lim_{z\to\xi} \frac{G_\lambda(y,z)}{G_\lambda(x,z)}\right)^2,$$

the fact that this limit exists is due to Ancona.

Proposition 1 (Ledrappier-Lim)**.** *g_t is rapid mixing with respect to μ_λ uniformly in λ.*

Idea of proof. We know that g_t is exponentially mixing with respect to Liouville measure, by work of Liverani. This would imply that g_t is topological power mixing, that is there exists $t_0, \alpha > 0$ such that for all $t > t_0$, $\frac{1}{r^\alpha}$, $\left(g_t(B(x,r)\right) \cap B(y,r) \neq \emptyset$. We need to prove that a uniform version of Dolgopyat's rapid mixing holds with respect to μ_λ, that is there exists c_0, c_1 independent of λ such that

$$\left|\int f h g_t \mu_\lambda - \int f h\right| \leq c_1 \|f\|_\alpha \|h\|)\alpha(1+t)^{-c_0}$$

□

We want to show that

$$\begin{aligned}\frac{\partial}{\partial\lambda} G_\lambda(x,y) &= \int_{\widetilde{M}} G_\lambda(x,z) G_\lambda(z,y)\,\mathrm{d}z \\ &= \int_0^\infty e^{RP(\lambda)} \int_{S(x,R)} \frac{G_\lambda(z,y)}{G_\lambda(z,x)} e^{-RP(\lambda)} G_\lambda^2(x,z)\,\mathrm{d}z\,\mathrm{d}R\end{aligned}$$

Recall that as $\lambda \to \lambda_0$,

$$\frac{G_\lambda(z,y)}{G_\lambda(z,x)} \to k_\lambda(x,y,\xi).$$

so,

$$\int_0^\infty e^{RP(\lambda)} \int_{S(x,R)} \frac{G_\lambda(z,y)}{G_\lambda(z,x)} e^{-RP(\lambda)} G_\lambda^2(x,z)\,\mathrm{d}z\,\mathrm{d}R = \int_0^\infty e^{RP(\lambda)} \int_{\partial\widetilde{M}} k_\lambda(x,y,\xi)\,\mathrm{d}\mu_x^{\lambda_0}(\xi)$$

Theorem 3 (Ledrappier-Lim)**.** $C(x,y) = \int_{\partial\widetilde{M}} k_{\lambda_0}(x,y,\xi)\, d\mu_x^{\lambda_0}(\xi)$

Remark 3. $\left\{\mu_x^{\lambda_0}\right\}$ minimizes the Mohsen-Rayleigh quotient.