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Thm (P). Let $\Gamma < PU(1,p) = \mathrm{Isom}(\mathbb{H}^p_{\mathbb{C}})$ be a lattice. Let $\rho : \Gamma \to PU(m,n)$ be a maximal representation. Assume $p>1$, $n>m$ and $\overline{\rho(\Gamma)}^{Z}$ has no factor of "tube type".

Then: $\rho = \rho_{std}\cdot \chi$

where: $\rho_{std} : \Gamma \hookrightarrow PU(1,p) \xrightarrow{s} PU(m, mp+k)$

$\chi : \Gamma \to Z_{PU(m,n)}(s(PU(1,p)))$ character.

PLAN. ① Rigidity Thm for maps between some parabolic geometries.

② Maximal representation.

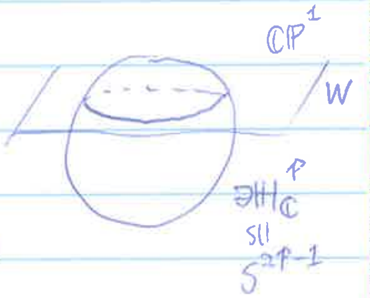
③ Convex hyperbolic lattices.

① Geometries $\qquad \mathbb{C}^{r,s} = (\mathbb{C}^{r+s}, h_{r,s})$ $h_{r,s}$ Hermitian of sign (r,s)

$$\partial\mathbb{H}^p_{\mathbb{C}} = \{ [x] \in \mathbb{CP}^p \mid h_{1,p}(x) = 0 \}$$

$$\mathbb{CP}^p \cong \mathbb{P}(\mathbb{C}^{1,p})$$

$$\overset{topo}{\cong} S^{2p-1} = PU(1,p)/Q$$



Lines: $W^{1,1} < \mathbb{C}^{1,p}$

$$C_W = \{ [x] \in \partial\mathbb{H}^p_{\mathbb{C}} \mid x < W^{1,1} \} \cong S^1$$

$\forall x,y \in \partial\mathbb{H}^p_{\mathbb{C}}$ $\exists!$ chain $C_{\langle x,y\rangle}$ containing them

Higher rank generalization

$$S_{m,n} = \{ W \in Gr_m(\mathbb{C}^{m,n}) \mid h_{m,n}|_W = 0 \}$$

m-chains $\quad$ Let $V^{m,m} < \mathbb{C}^{m,n}$ subspace

$$C_V = \{ W \in S_{m,n} \mid W < V \}$$

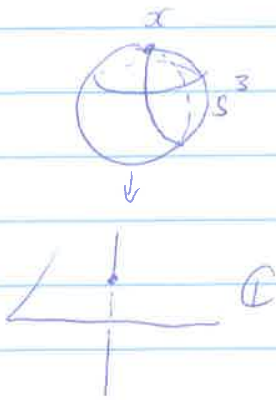
Rk: $\forall x,y \in S_{m,n}$, $x \pitchfork y$, there exists a unique m-chain $C_{\langle x,y\rangle}$ containing them.

Rk: $(x,y,z) \in S^3_{m,n}$ pairwise transversal, then $\exists$ an m-chain containing them $\dim_{\mathbb{C}} \langle x, y, z \rangle < 2n$.

Thm 2 Let $\varphi : \partial \mathbb{H}^p_{\mathbb{C}} \longrightarrow S_{m,n}$ measurable map, Zariski dense,
assume $\forall (x,y,z) < \partial \mathbb{H}^p_{\mathbb{C}}$ with $\langle x,y,z \rangle = 2$
Then $\dim_{\mathbb{C}} \langle \varphi(x), \varphi(y), \varphi(z) \rangle = 2m$
Then $\varphi =_{a.e.} \overline{\varphi}$ (real) algebraic map.

PROOF: $p=2$ $m=1$ $n=2$. $\varphi|_{Chain}$ is rational.

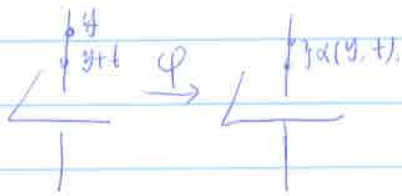
Fact 1. $\partial \mathbb{H}^2_{\mathbb{C}} \setminus \{x\} = \mathbb{C} \times \mathbb{R} = \mathrm{Heis}$
and in this model chains through x are vertical lines.



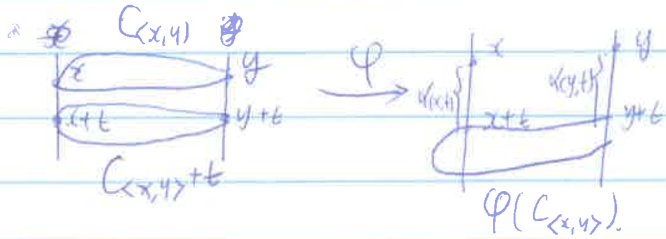
$r : \partial \mathbb{H}^2_{\mathbb{C}} \setminus \{x\} \longrightarrow \mathbb{C}$ corresponds to associate to y the chain $C_{\langle x, y \rangle}$

Fact 2 Chains that do not contain x are mapped under r to Euclidean circles. and $\forall$ Euclidean circle Q and any y.
$\exists !$ Chain C through y projecting to Q

$\varphi : \partial \mathbb{H}^2_{\mathbb{C}} \longrightarrow \partial \mathbb{H}^2_{\mathbb{C}}$ assume $\varphi(x) = x$
~~since~~ since φ preserves the geometry gives $\varphi_x : \mathbb{C} \to \mathbb{C}$
$\alpha : \partial \mathbb{H}^2_{\mathbb{C}} \setminus \{x\} \times \mathbb{R} \longrightarrow \mathbb{R}$
$(y, t) \longmapsto \varphi(y) - \varphi(y+t)$



Enough to show that α doesn't depend on y.



$\Rightarrow \alpha(y,t) = \alpha(x,t)$.

② Maximal representation

Teichmüller space of Σ_g

$\{$marked hyperbolic structure$\}/$isotopy $\longrightarrow \mathrm{Hom}(\Gamma_g, PSL_2\mathbb{R})/\!\!/PSL_2\mathbb{R}$.

<u>Thm</u> (Goldman) $\rho \in \mathrm{Hom}(\Gamma, PSL_2\mathbb{R})$ is the holonomy of a hyperbolisation iff $|E(\rho)| = 2g-2$.

Take $F : \tilde{\Sigma}_g \longrightarrow \mathbb{H}^2$ that is ρ-equivariant. $\omega \in \Omega^2(\mathbb{H}^2)^{PSL_2\mathbb{R}}$

$$E(\rho) = \frac{1}{2\pi}\int_\Sigma F^*\omega$$

$$\Omega^2(\mathbb{H}^2)^{PSL_2\mathbb{R}} \ni \omega$$

$$H_b^2(\Sigma, \partial\Sigma, \mathbb{R}) \simeq H_b^2(\Gamma, \mathbb{R}) \xleftarrow{\rho^*} H_{cb}^2(PSL_2\mathbb{R}, \mathbb{R}) \ni k_b$$

$$T_\rho = \langle \rho^* k_b, [\Sigma, \partial\Sigma] \rangle$$

~~Def. Let G be an~~

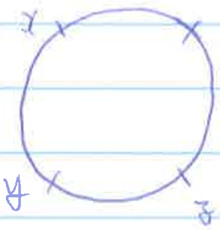
G simple Lie group non-compact

G/K symmetric space.

Hermitian iff $\exists$ G-invariant complexe structure
iff $\Omega^2(G/K)^G = \mathbb{R}$.

Let G be Hermitian, $\Gamma = \pi_1(\Sigma)$.
$\rho: \Gamma \to G$ is maximal if $\langle \rho^* \kappa_b^G, [\Sigma, \partial\Sigma] \rangle = \mathrm{rk}(G) \chi(\Sigma)$
$\leq \mathrm{rk}(G)\, \chi(\Sigma)$
Toledo

Thm (Burger – Iozzi – Wienhard)
$\rho: \Gamma \to SU(m,n)$ is maximal iff $- \rho(\Gamma) \subset SU(m,m)$
$- \exists \varphi: \partial\Gamma \to S_{m,m}$ (right) continuous
ρ-equivariant and "positive"



$(\varphi(x), \varphi(y), \varphi(z)) \longrightarrow C \in Her(\mathbb{C}^m)$
$\Uparrow$
$S_{m,m}$.

③ Complex Hyperbolic Lattices
$\Gamma < PU(1,p) = \mathrm{Isom}(\mathbb{H}^p_{\mathbb{C}})$
e.g. if $p = 1$ $\mathbb{H}^1_{\mathbb{C}} \cong \mathbb{H}^2_{\mathbb{R}}$.
$\rho: \Gamma \to G$ $\quad G$ Hermitian
$\Updownarrow$
$\exists$ Kähler form ω_G.

$\rho^* \omega_G \in H^2_b(\Gamma, \mathbb{R})$.
$T(\rho) = \frac{1}{p!} \int_{\Gamma \backslash \mathbb{H}^p_{\mathbb{C}}} \rho^* \omega_G \wedge \omega^{p-1}_{\mathbb{H}^p_{\mathbb{C}}}$ $\qquad |T(\rho)| \leq \mathrm{rk}\, G \cdot \mathrm{Vol}(\Gamma \backslash \mathbb{H}^p_{\mathbb{C}})$

FACT Let $\rho : \Gamma \to PU(m,n)$ be maximal
$\varphi : \partial \mathbb{H}^2_{\mathbb{C}} \longrightarrow S_{m,n}$ be ρ-equivariant
then ρ satisfies the hypothesis of thm 2.

PROOF THM 1 – You can assume that $\rho(\Gamma)$ is Zariski dense.
$\rightarrow$ you get a measurable ρ-equivariant boundary map.

Corollary 1: No Zariski dense maximal representations
$\rho : \Gamma \longrightarrow PU(m,n)$ if $p > 1$ $n > m > 1$.

Corollary 2: If $p > m^2$, then ρ is $\rho_{std} \circ \chi$

tube type: the associated linear reps $\rho : \Gamma \to GL(\mathbb{C}^{m,n})$ has no irreducible subspace on which h has signature (k,k).

Corollary 3: The representation $\rho_{std} \cdot \chi$ is locally rigid.