Asymptotics of certain families of Higgs bundles in Hitchin compact

\[ \text{Hom}(\text{PSL}(n, \mathbb{C}), \text{PSL}(n, \mathbb{R}))/\text{PSL}(n, \mathbb{R}) \]

\[ \text{Hit}_n \xrightarrow{\sim} \bigoplus_{i=1}^n H^0(\Sigma, K^{\otimes i}) \]

Higgs bundle.
A system of equations.

Goal: Understanding the asymptotic geometry of \( \psi_t \) in terms of
asym of subloci of \( \bigoplus_{i=2}^n \mathcal{H}(\Sigma, K^i) \).
Closely related to Pandit's talk
(\( \overline{M}_{\text{DR}} \leftrightarrow \overline{M}_{\text{Betti}} \) (\( \overline{M}_{\text{Higgs}} \leftrightarrow \overline{M}_{\text{Betti}} \))

For example: \( t \to \infty \) a family of \( \psi_t \)-equiv. harm maps \( f_t: \Sigma \to \text{SL}(n, \mathbb{C}) \)
(\( f_t: \Sigma \to \text{PSL}(n, \mathbb{C})/\text{SO}(n) \))
As \( t \to \infty \), \( f_t(U) \) becomes more and more flat!

Fix \( \Sigma \) a Riemann surface.

Def: A \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle over \( \Sigma \) is \( (\Sigma, \phi) \) where
\( \Sigma \) a rank \( n \) holo bundle over \( \Sigma \) with \( \det \Sigma = \Theta \\
\phi \in \mathcal{H}^0(\Sigma, \text{End}(\Sigma) \otimes K) \) with \( \text{tr} \phi = 0 \\
\phi: \Sigma \to \Sigma \otimes K \)
Def. A $SL(n, \mathbb{R})$-Higgs bundle is $(E, \phi, \Omega)$
- $(E, \phi)$ is $SL(n, \mathbb{C})$-Higgs bundle
- $\Omega : E \times E \to \mathbb{R}$, nondegenerate.
- $\phi$ is $\Omega$ symmetric i.e. $\phi^* \Omega = \Omega \phi$

Ex. $SL(Z, \mathbb{R})$
$$E = \mathbb{K}^2 \oplus \mathbb{K}^{-2}, \quad \phi = \begin{pmatrix} 0 & \mathbb{K}_2 \\ \mathbb{K}_2 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Imrn (rk 2 by Hitchin, general by Simpson)
Let $(E, \phi)$ be a stable Higgs bundle, then $E$! Hermitian metric $h$ s.t.
$$\left( \ast \right) \quad F_h + [\phi, \phi^* h] = 0$$
Curvature of Chern connection.

Parameterization (Hitchin fiberation and section)
$$\mathfrak{s}_{SL(n, \mathbb{R})} \rightarrow (E, \phi) \rightarrow \mathbb{C}^n$$

$$(p(\Phi), \ldots, p(\Phi)) \in \bigoplus_{j=2}^n H^0(\mathbb{Z}, K^2)$$
$P_i(\Phi)$ is homogeneous $SL(n, \mathbb{C})$-ad invariant poly on $sl(n, \mathbb{C})$.

E.g. $P_i(\Phi) = \text{tr} \Phi^2$

- Imrn $(S)$ is onto a component of $SL(n, \mathbb{R})$-Higgs bundles
$$(E, \phi, \Omega) = (E, \phi, \Omega).$$
$$E = \mathbb{K}^{g_1} \oplus \ldots \oplus \mathbb{K}^{g_n}, \quad \phi = \begin{pmatrix} 0 & \mathbb{K}_{g_2} & \mathbb{K}_{g_3} & \cdots & \mathbb{K}_{g_n} \\ \mathbb{K}_{g_2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbb{K}_{g_n} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Q: Consider \((E, \Phi_t) \in \text{Hit}_n, \ t \in \mathbb{R}^+ \rightarrow \infty\)

Solving Hitchin eqn \(\rightarrow \) \(h_t\) on \(E\)

\[ \nabla_{h_t} = \nabla_{h_t} + \Phi + \Phi^* h \] flat conexions on \(E\)

\[ T_h(t) = E_{\sigma(0)} \rightarrow E_{\sigma(L)} \quad \sigma(5i) \in \text{sc}[0, L] \]

\[ T_{\theta}(t) \cdot F \quad \sim \sim \quad \Phi_t : \Pi_1(\mathbb{Z}) \rightarrow \mathfrak{sl}(n, \mathbb{R}) \]

parallel transport.

\[ \nabla_{h_t} + h_t \rightarrow \text{equiv hom maps} \quad \Phi_t : \mathbb{Z} \rightarrow \mathfrak{sl}(n, \mathbb{R})/\mathfrak{so}(n) \]

As \(t \rightarrow \infty\), what is \(h_t, \nabla_t, T_h(t), \Phi_t\)?

We restrict to \(\Phi_t = \begin{pmatrix} 0 & -t \mathbb{I}_n \\ 0 & 0 \end{pmatrix} \) or \(\begin{pmatrix} 0 & t \mathbb{I}_n \\ 0 & 0 \end{pmatrix} \)

**Prop. (Benaglia, Cellier)**

\(h_t\) is diagonal.

**Thm. (Cellier, -L)**

The solution \(h_t\) to (\(*\)):

\[ h_t = \text{diag}(l_1 t \mathbb{I}_n, l_2 t \mathbb{I}_n, \ldots, l_n t \mathbb{I}_n) \left(1 + O(t^{-2})\right) \]

true for any nonzero pt \(\mathfrak{g}_n\) of \(\mathfrak{g}_n\).
For any \( t \neq 0 \), \( \gamma \) is a geodesic \( \gamma(s) = e^{\frac{s}{2}} \) not passing through zeros of \( q_n \).

If \( \gamma \) is not heading close to zeros, then \( T_\gamma(\theta) = U e^{i t \text{Re} \bar{q}_n} U^{-1} (Id + O(t^{-1})) \).

Geometry:
\[ q_n = d^2 \]

Locally, \( q_n \) defines \( n \) foliations with direction of measure
\[ F_2, \ldots, F_n \]

\( n = 3 \)

Cor.: As \( t \to \infty \), Hitchin eqn. \( F_{\gamma^k} + [\phi, \phi^k] = 0 \) decouples:

\[ [\phi, \phi^k] = o(1) \]

i.e., \( \nabla_{nt} \) is going to be flat.

\( SL(2, \mathbb{C}) \): Taubes, Mazzeo - Suoboda - Weiss - Witt.

\( Rk. \): (0, 93) case is proved by Loftin.

Our methods by Loftin's work and Dumas - Wolf.
Cor. (hurm map $\Phi$)

$V\Phi$ not zero of $\mathfrak{g}_n$. If a nbhd llp of $\Phi$ s.t.

$\Phi_t(\mathcal{U})$ becomes more and more flat.

$\rightarrow f_t: \Sigma \rightarrow (SL(n,\mathbb{R})/SO(n), \frac{1}{t}d)$

$f_{\mathcal{U}}: \Sigma \rightarrow \mathcal{O}_{\mathcal{C}} \mathcal{O}_{\mathcal{C}}$

$f_{\mathcal{U}}(V\Phi)$ is inside a 2-plane in an appartment of $\mathcal{O}_{\mathcal{C}}$

**Sketch of proof for $h^{-}$**

$E = K\frac{\partial_1}{\partial z} + \ldots + K\frac{\partial_n}{\partial z}$

$\phi = \begin{pmatrix} e^{\lambda^1} \\ \vdots \\ e^{\lambda^n} \end{pmatrix}$

$h = \begin{pmatrix} e^{\lambda^1} \\ \vdots \\ e^{\lambda^n} \end{pmatrix}$

$\Phi^{-} + [\Phi, \Phi^{+}] = 0.$

$\Phi^{-} \Phi^{+}$

$\mathcal{F}(K^{-} \mathcal{C} h) \rightarrow$

$\left\{ \begin{array}{l}
\lambda_3^1 + e^{-2\lambda^1} |e_{1n}|^2 - e^{\lambda^1 - \lambda^2} = 0 \\
\lambda_3^2 + e^{\lambda^1 - \lambda^2} - e^{\lambda^2 - \lambda^3} = 0 \\
\vdots \\
\lambda_3^n + e^{\lambda^1 - \lambda^2} - e^{\lambda^2 - \lambda^3} = 0 \\
\end{array} \right.$

Consider a base $g$ s.t.

$g = |e_{1n}|^2$ on $K \subset \mathcal{C}$

$|e_{1n}|^2 \leq e^{\lambda^1 - \lambda^2}$ outside $K$

Define $U = \lambda^1 - \frac{n-1}{2} \ln g$, a function.
then \[
\Delta u^3 = -e^{u^2}u^3 + e^{u^2}u^2 + \frac{n^5}{4}\text{ kg}.
\]

**Maximum Principle (To show)**

For example, at \(\max u^3\),
\[
0 \geq \Delta g u^3 = -e^{u^2}u^3 + e^{u^2}u^2 + \frac{n^5}{4}\text{ kg}
\]
\[
\Rightarrow e^{u^2}u^3 \geq e^{u^2}u^2 + \frac{n^5}{4}\text{ kg}
\]
\[
\Rightarrow \max e^{u^2}u^3 \geq \max e^{u^2}u^2 - C \quad (\text{come from the sign in } \Theta)
\]

**Sketch of Proof for \(\text{To}(t)\)**

\[
D = \nabla u + \phi + \phi \nabla \phi.
\]

\[
= \begin{pmatrix} n_3 \nabla u & t \phi \\
-\phi & \nabla \phi \end{pmatrix} dS + \begin{pmatrix} \phi \nabla \phi \nabla u \\
-\phi \nabla \phi & \nabla \phi \end{pmatrix} dS
\]

**Restrict \(D\) on the path \(\tau(s) = s e^{i\theta}, s \in [0, L]\)**

Solving \(\text{To}(t) \iff D_{\text{To}(5)} \Phi(s) = 0\)

\[
\Phi(s) = \begin{pmatrix} e^{i\theta} \frac{t}{e^{i\theta}} e^{i\theta} e^{i\theta} \\
\frac{t}{e^{i\theta}} e^{i\theta} e^{i\theta} + R \end{pmatrix} \Phi(s)
\]