Jonathan Mattingly Lecture 1

$\{u_t : t \geq 0\}$ $u_t \in X$ ← Polish space

Σ - a σ-algebra

t here may be cts or discrete.

Markov Processes $P(u_0, A) = \mathbb{P}(u_1 \in A \mid u_0)$

↑ prob u_1 is in space A given initial data u_0.

$P : X \times \Sigma \to [0,1]$

Markov transition kernel

$x \mapsto P(x, \cdot)$ is a probability measure

$A \mapsto P(\cdot, A)$ is a measurable function

$\varphi : X \to \mathbb{R}$ $(P\varphi)(x) = \int_X \varphi(y) P(x, dy)$

↳ some real-valued statistic of state space

P acts on φ

If μ is a probability measure $\mu \in M(X)$ then define $(\mu P)(A) := \int P(x, A) \mu(dx)$, $A \in \Sigma$

Let $X = \{1, \ldots, M\}$. Then $P_{ij} = P(u_1 = j \mid u_0 = i)$

$\varphi = \begin{pmatrix} \varphi(1) \\ \vdots \\ \varphi(m) \end{pmatrix}$, $P\varphi$ ← just matrix multiplication, $\mu = (\mu(1), \ldots, \mu(m))$

μP ← left multi of matrix

$P_m := \underbrace{P \cdots P}_{m \text{ times}}$, $P_{n+m} = P_m P_n = P_n P_m$

$P_t \varphi \doteq \mathbb{E}_{u_0 = x} \varphi(u_t) = \mathbb{E}(\varphi(u_t) \mid u_0 = x)$

μ is invariant (stationary) if $\mu P = \mu$

alternatively $\mu P f = \mu f$ for all ~~bdd~~ bdd cts f

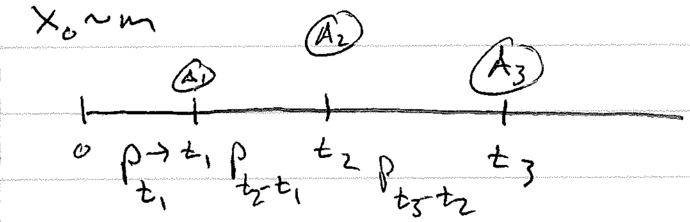
$X^{\mathbb{N}} := \{ (x_0, x_1, \ldots) : x_i \in X,\ i \in \mathbb{N} \}$

$x = (x_0, x_1, \ldots) \in X^{\mathbb{N}}$. Let m be a measure on X.

Define $mP_{\mathbb{N}}$ as a measure on $X^{\mathbb{N}}$.

$\hookrightarrow$ done through the Kolmogorov ext. thm.

$x_0 \sim m$



What's the prob of being in A_1 at t_1, A_2 at time t_2, and A_3 at time t_3? What is the measure of this set?

$$mP_{\mathbb{N}}(A) = \int_{X \times A_1 \times A_2 \times A_3} m(dx_0) P_{t_1}(x_0, dy_1) \cdot P_{t_2 - t_1}(y_1, dy_2) \cdot P_{t_3 - t_2}(y_2, dy_3)$$

Now we define a shift operator:

$\theta : (x_0, x_1, \ldots) \longmapsto (x_1, x_2, \ldots)$

Suppose $\theta : X^{\mathbb{N}} \to X^{\mathbb{N}}$. a measure M on $X^{\mathbb{N}}$ is invariant if $M\theta = M$ ($M(\theta^{-1}(A)) = M(A)$ $\forall A$

$$\begin{cases} f : X^{\mathbb{N}} \to \mathbb{R} \text{ is invariant if} \\ f\theta^{-1} \text{ is inv a.s. } \hookrightarrow \text{wrt } M \\ \text{a set } A,\ \theta^{-1}(A) = A \text{ a.s.} \end{cases}$$

Let $(X^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \theta, M)$, $\Phi : X^{\mathbb{N}} \to \mathbb{R}$. Then

$\uparrow$ invariant for θ

$M = \mu P_{\mathbb{N}}$ (μ: inv measure for P)

1) $\frac{1}{N} \sum_{i=0}^{N-1} \varphi(\theta^i x) \xrightarrow[M\text{-a.s.}]{} \bar{\varphi}(x)$

$\quad \parallel$

" $\frac{1}{N} \sum_{i=0}^{N-1} \varphi(X_i)$ " $\qquad \Phi(x) = \varphi(x_0)$

2) $\bar{\Phi}$ is Θ inv and $\int \Phi(y) M(dy) = \int \bar{\Phi}(y) M(dy)$

a measure M is ergodic if for any invariant set A, $M(A) \in \{0,1\}$.

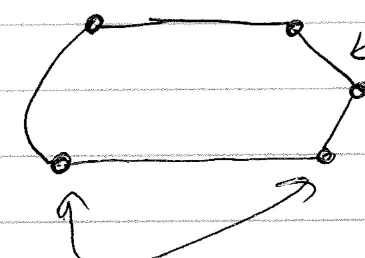
M is ergodic $\iff$ any shift inv Φ is M-a.s. const.

$$\iff \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} m(A \cap \Theta^{-n} B) = m(A) m(B) \quad \forall A, B$$

Notice $\frac{1}{N} \sum_{n=0}^{N-1} m(A \cap \Theta^{-n} B)$ is $\mathbb{P}(X_0 \in A, X_n \in B)$ if A, B are $A \times X \times X \cdots$

ie a cylinder set

Any invariant measure M is mixture of ergodic invariant measures.



space of inv measures

note convex since if m_1, m_0 are inv then $m_\alpha = \alpha m_1 + (1-\alpha) m_0$ is inv

extremal pts are ergodic measures.

(every other set can be written as linear combination.

Let m, ν be inv measures

1) if m is ergodic and $\nu \ll m \Rightarrow m = \nu$

2) if m, ν are ergodic, then $m = \nu$ or $m \perp \nu$

3) m is an extremal pt iff m is ergodic

A set A is inv for Markov operator P with inv measure μ $(\mu P = \mu)$ if

$$P(x, A) = 1 \quad \mu\text{-a.e. } x \in A$$
$$P(x, A) = 0 \quad \mu\text{-a.e. } x \in A^c$$

μ is ergodic if every inv set A of P with μ inv satisfies $\mu(A) \in \{0, 1\}$.

μ is ergodic $\Leftrightarrow$ $\mu P_{\mathbb{N}}$ is ergodic

$$\begin{array}{ccc} \mu \text{ ergodic} & \longleftrightarrow & \mu \text{ extremal} \\ \updownarrow & & \updownarrow \\ \mu P_{\mathbb{N}} \text{ ergodic} & \longleftrightarrow & \mu P_{\mathbb{N}} \text{ is extrema} \end{array}$$

$X = \{1, \dots, N\}$

Suppose

$\forall j$, $\inf_i P_{ij} \geq c_j \geq 0$

$\sum c_j > 0$

$\Rightarrow \exists!$ inv measure

$\nu = \frac{1}{c} \sum_j c_j \delta_j$

We will prove this next time

$X = [0, 1]^{\mathbb{N}}$

Suppose

$\inf_{x \in X} P(x, \cdot) \geq \alpha \nu(\cdot)$

ν a prob measure

$\Rightarrow \exists!$ inv measure