Kenji Nakanishi Lecture 1

Goal: Classify and predict behaviors of solns to nonlinear dispersive PDE.

(NLS): $i\dot{u} - \Delta u + Vu = |u|^2 u$

$u(x,t)$, $(t,x) \in \mathbb{R}^{1+3} := \mathbb{R} \times \mathbb{R}^3$

we consider radial soln $u(t,x) = u(t,|x|) \in \mathbb{C}$

$V = V(|x|) \in \mathcal{S}(\mathbb{R}^3; \mathbb{R})$

potential schwartz class

We consider four types of solns and our goal is to classify solns based on initial data.

Outline: (1) Introduce problem and present results

(2), (3) Proof (ideas)

First we will assume $V=0$

Then $V \neq 0$ if we have time.

Problem:

(NLS) LWP in $H^1_r = \{\varphi \in H^1(\mathbb{R}^3) \mid \varphi = \varphi(|x|)\}$

For any initial data $u(0) \in H^1_r$, $\exists!$ u soln in $C(I; H^1_r)$ where I is a maximal interval.

Also cts dependence on initial data.

We can estimate the existence interval: $|I| \gtrsim \|u_0\|_{H^1}^{-1/4}$

We want to predict $u(t)$ from $u(0)$

Energy and Mass:

$$E(u) = \int_{\mathbb{R}^3} \frac{|\nabla u|^2}{2} + \frac{V|u|^2}{2} - \frac{|u|^4}{4}\,dx$$

focusing

$$M(u) = \int_{\mathbb{R}^3} \frac{|u|^2}{2}\,dx$$

(NLS) $\Rightarrow$ E, M are conserved in time

(NLS) $\Leftrightarrow$ $\dot{u} = iE(u)$

Sympletic form $\langle if | g \rangle$ where

$$\langle f | g \rangle = \mathrm{Re} \int_{\mathbb{R}^3} f(x)\overline{g(x)}\,dx$$

Typical Solns:

dispersion dominates then scattering

For NLS, $\exists v$ soln to $i\dot{v} - \Delta v = 0$ st $\|u(t) - v(t)\|_{H^1} \xrightarrow{t\to\infty} 0$.

nonlinearity dominates then blowup

For NLS, $\exists T < \infty$ st $\|u(t)\|_{H^1} \xrightarrow{t\to T} \infty$

balance then Solitons

$$u(t,x) = e^{-it\omega}\varphi(x) \qquad \omega > 0$$

For NLS, $0 = E'(\varphi) + \omega M'(\varphi)$

(SE) $0 = -\Delta\varphi + V\varphi + \omega\varphi - |\varphi|^2\varphi$

For Solitons, two types $\begin{cases} \text{Stable} \longrightarrow \text{asymptotic profile} \\ \text{unstable} \longrightarrow \text{threshold solns} \end{cases}$

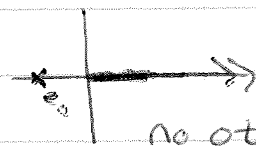
Stable $\Leftrightarrow$ $\forall \varepsilon > 0,\ \exists \delta > 0$ st $\operatorname{dist}_{H^1}(u(0), \{e^{i\theta}\varphi\}_\theta) < \delta$

$\Rightarrow \forall t \in \mathbb{R},\ \operatorname{dist}_{H^1}(u(t), \{e^{i\theta}\varphi\}_\theta) < \varepsilon$

<u>Assume for V:</u>

$H := -\Delta + V$ $\qquad H\phi_0 = e_0 \phi_0$

spec(H)



$0 < \phi_0 \in H^1_r$

$\|\phi_0\|_2 = 1$

<u>Solitons:</u>

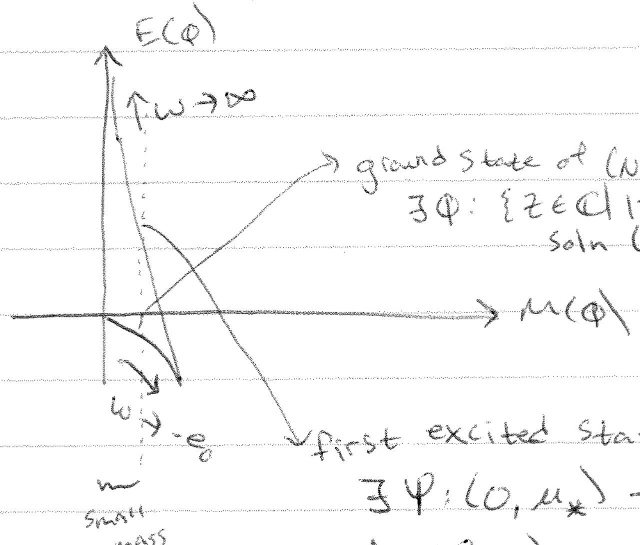
We only consider ground states of (SE)

$\forall \omega > -e_0$, $\exists \phi_\omega$ Soln of (SE) st

$E_\omega = E + \omega M$ and $E_\omega(\phi_\omega) = \inf\{E_\omega(\phi) \mid \phi \neq 0,\ \phi$ Soln of (SE)$\}$

$= \inf_{\phi \neq 0} \max_{\lambda > 0} E_\omega(\lambda\phi)$

Mass vs Energy of Soliton



$\to$ ground state of (NLS) $\to$ Stable Solitons

$\exists \Phi : \{z \in \mathbb{C} \mid |z| < z_*\} \to H^1_r$

Soln (SE), $0 < z_* \ll 1$ a const

$\Phi(z) = z\phi_0 + O(z^3)$

$\omega = -e_0 + O(z^2)$

$\to$ first excited stat of (NLS) $\to$ unstable Solitons

$\exists \Psi : (0, \mu_*) \to H^1_r$ Soln (SE), $0 < \mu_* \ll 1$

$M(\Psi[\mu]) = \mu$, $\omega = \mu^{-2} M(Q)^2 + O(1)$

Q: ground state of (NLS, $V = 0$)

$-\Delta Q + Q = Q^3$, $0 < Q \in H^1_r$

$$\Psi = \omega^{1/2}\left(Q + O(\omega^{-1})\right)(\omega^{1/2} x)$$

$$E_x(\mu) = E(\Psi[\mu])$$

Scaling factor

For $0 < \varepsilon \ll 1$

$$H(\varepsilon) = \{ \phi \in H^1_r \mid M(\phi) = \mu < \mu_*,\ E(\phi) < E_x(\mu) + \mu^{-1}\varepsilon^2 \}$$

Classification of initial data

"Scattering to Φ"

$$\mathcal{S} := \{ u(0) \in H^1_r \mid \exists z : [0,\infty) \to \mathbb{C},\ |z| < z_*$$

$$\exists v : \text{free sol. } \left(\substack{\text{solves} \\ iv - \Delta v = 0}\right)$$

$$\| u(t) - \Phi[z(t)] - v(t) \|_{H^1} \xrightarrow{t \to \infty} 0 \}$$

"Blowup"

$$\mathbb{B} := \{ u(0) \in H^1_r \mid \exists T < \infty,\ \| u(t) \|_{H^1} \xrightarrow{t \to T} \infty \}$$

"Trapped by Ψ"

$$\mathcal{J} := \{ u(0) \in H^1_r \mid \overline{\lim_{t \to \infty}}\ \operatorname{dist}_{H^1_\mu} \left(u(t), \{ e^{i\theta} \Psi[\mu] \}\right) \lesssim \varepsilon \}$$

$$M(u) = \mu$$

$$\mathcal{S} \to \overline{\lim_{t \to \infty}}\ \| u(t) \|_4^4 \le \sup_{M(\Phi) \le M(u)} \| \Phi[z] \|_4^4 \simeq \mu \ll 1$$

$$\mathcal{J} \to \underline{\lim}_{t \to \infty}\ \| u(t) \|_4^4 \ge \| \Psi[\mu] \|_4^4 - C\varepsilon \gtrsim \mu^{-1} \gg 1$$

$\mathcal{S}, \mathbb{B}, \mathcal{J}$ are mutually disjoint.

Dynamics for $t < 0$, $u(t)$ soln $\to$ $\bar{u}(-t)$ soln

Thm: If $0 < \mu_*$, $\varepsilon \ll 1$, then

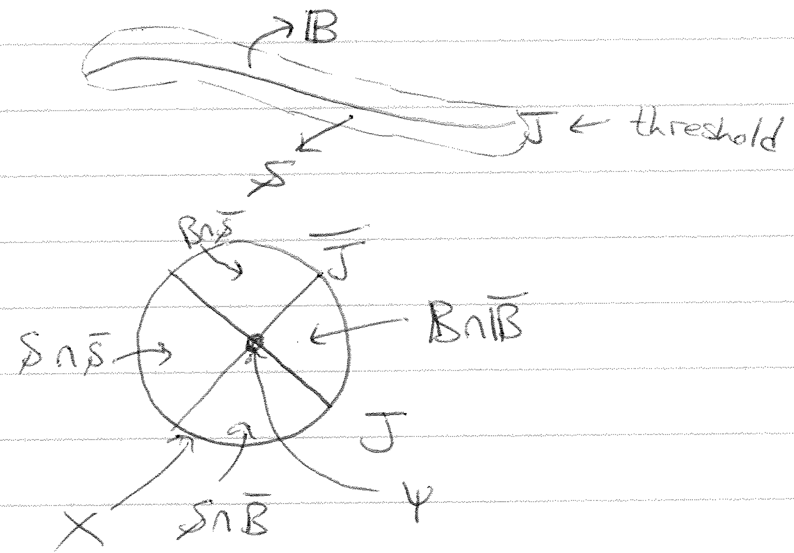
$\mathcal{H}(\varepsilon) \subset \mathbb{S} \cup \mathbb{B} \cup \mathbb{J}$. Moreover,

$\mathbb{J}$: C^1-mfd $\subset H^1_r$, codim=1, unbdd on $M=\mu$.

$\mathbb{J} \cap \bar{\mathbb{J}}$: C^1-mfd codim=2 bdd,

$O(\varepsilon)$ neighborhood of $\{e^{i\theta}\Psi[\mu]\}$ in H^1_r

for $0<\mu<\mu_*$



$\exists X$: $O(\varepsilon)$ neighborhood of Ψ. $X = \bar{X}$

(1) ~~H(ε)~~ $\forall u(0) \in \mathcal{H}(\varepsilon)$,

$u^{-1}(X)$ is an interval

(2) $(\mathbb{S}\cap\bar{\mathbb{B}}) \cup (\mathbb{B}\cup\bar{\mathbb{S}}) \ni \forall u(0)$,

$u^{-1}(X)$ is non-empty compact

(3) $u(0) \in \mathbb{J} \iff u^{-1}(X)$ is a neighborhood of ∞.