Invariant measures for nonlinear PDE

Andrea R. Nahmod

University of Massachusetts Amherst

August 27th-28th, 2015

Introductory Workshop at MSRI

Derivative NLS Equation on $\mathbb{T}$

In this talk we consider the periodic DNLS and study the existence of invariant measures and a.s global well posedness.

(DNLS)
$$\begin{cases} u_t - i\, u_{xx} = (|u|^2 u)_x \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}. \end{cases}$$

This is a Hamiltonian PDE which is completely integrable. In particular:

- Mass: $m(u) = \frac{1}{2\pi} \int_{\mathbb{T}} |u(x,t)|^2 \, dx$
- 'Energy': $E(u) = \int_{\mathbb{T}} |u_x|^2 \, dx + \frac{3}{2} \mathrm{Im} \int_{\mathbb{T}} u^2 \overline{u u_x} \, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6 \, dx$
- Hamiltonian: $H(u) = \mathrm{Im} \int_{\mathbb{T}} u \bar{u}_x \, dx + \frac{1}{2} \int_{\mathbb{T}} |u|^4 \, dx$ (at $\dot{H}^{\frac{1}{2}}$ level).

are conserved quantities of time.

In looking for solutions to (DNLS) we face a derivative loss arising from the nonlinear term and hence for low regularity data the **key** is to somehow make up for this loss.

- On $\mathbb{R}$: the equation is scale invariant for data in L^2, that is $s_c = 0$.
  - LWP in $H^{\frac{1}{2}}(\mathbb{R})$ by Takaoka (99').
  - GWP in $H^s(\mathbb{R}), s > \frac{1}{2}$ with small L^2-norm (so that energy is positive by Gagliardo-Nirenberg) by Colliander-Keel-Staffilani-Takaoka-Tao (02').
  - There is some form[1] of ill-posedness for data in $H^\sigma(\mathbb{R}), \sigma < 1/2$.

[1] data-sol map fails to be C^3 or uniformly C^0 (Biagioni-Linares).

Deterministic Local Theory for DNLS on $\mathbb{T}$

- S. Herr (06') showed LWP for initial data $u(0) \in H^\sigma(\mathbb{T})$, if $\sigma \geq \frac{1}{2}$.
  - GWP for $\sigma \geq 1$ and small L^2 data; also in $H^\sigma(\mathbb{T})$ for $\sigma > 1/2$ also holds (I-method).)

- A. Grünrock and S. Herr (08') showed showed LWP for initial data $u_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ and $2 \leq r < 4$, $s \geq 1/2$.

$$\|u_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \| \langle n \rangle^s \hat{u}_0 \|_{\ell^r_n(\mathbb{Z})} \qquad r \geq 2$$

These spaces scale like Sobolev $H^\sigma(\mathbb{T})$, with $\boxed{\sigma = s + 1/r - 1/2}$.

For example for $s = 2/3-$ and $r = 3$ $\quad \sigma < 1/2$.

All LWP results rely on studying an associated **gauged equation**.

Periodic Gauged Derivative NLS Equation

- Why do we need to gauge? Because the nonlinearity:

$$(|u|^2 u)_x = u^2\, \overline{u}_x + 2\, |u|^2\, u_x \quad \text{hard to control.}$$

Periodic Gauge Transformation (Herr, 06): For $f \in L^2(\mathbb{T})$

$$G(f)(x) := \exp(-iJ(f)(x))\, f(x)$$

where

$$J(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \int_\theta^x \left(|f(y)|^2 - \frac{1}{2\pi} \|f\|^2_{L^2(\mathbb{T})} \right) dy\, d\theta$$

is the **unique** 2π**-periodic mean zero primitive** of the map[2]

$$x \longrightarrow |f(x)|^2 - \frac{1}{2\pi} \|f\|^2_{L^2(\mathbb{T})}.$$

Then, for $u \in C([-T, T]; L^2(\mathbb{T}))$ the (adapted) periodic gauge is defined as

$$\mathcal{G}(u)(t, x) := G(u(t))(x - 2\, t\, m(u))$$

[2] $G(f)$ is 2π-periodic since integrand has zero mean value.

- $\mathcal{G}: C([-T,T]; H^\sigma(\mathbb{T})) \to C([-T,T]; H^\sigma(\mathbb{T}))$ is a homeomorphism.
- $\mathcal{G}$ is locally bi-Lipschitz on subsets in $C([-T,T]; H^\sigma(\mathbb{T}))$ with prescribed L^2-norm.
- The same is true if we replace $H^\sigma(\mathbb{T})$ by $\mathcal{F}L^{s,r}$, the Fourier-Lebesgue spaces.

What is the gauged DNLS equation?

If u is a solution to DNLS and $v := \mathcal{G}(u)$ we have that v solves:

$$\text{(GDNLS)} \qquad v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v$$

with initial data $v(0) = \mathcal{G}(u(0))$ and where

$$m(u) = m(v) := \frac{1}{2\pi}\int_{\mathbb{T}} |v|^2(x,t)dx = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2(x)dx$$

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v\overline{v}_x)\,dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4 dx - m(v)^2$$

Note both $m(v)$ and $\psi(v)(t)$ are real.

Local well-posedness for (GDNLS) in H^σ gives local existence and uniqueness for DNLS in H^σ; but don't necessarily have all the auxiliary estimates coming from the fixed point argument used to obtain LWP for (GDNLS).

Invariant weighted Wiener measures and a.s GWP

Since by Grünrock-Herr's result we have have LWP at a 'level of regularity' below $\frac{1}{2}$, we can start thinking about weighted Wiener measures for the periodic DNLS, constructed from the energy conservation $E(u)$.

- **Goal 1**: Construct an associated invariant weighted Wiener measure and establish GWP for data living in its support. In particular almost surely for data living in a Fourier-Lebesgue space scaling like $H^{\frac{1}{2}-}(\mathbb{T})$
(A.N–, T. Oh, L. Rey-Bellet, G. Staffilani).

- **Goal 2**: Show that the ungauged invariant Wiener measure associated DNLS obtained above is absolutely continuous with respect to the weighted Wiener measure for DNLS constructed by Thomann and Tzvetkov directly. In particular we thus prove the invariance of the latter. We prove a general result on absolute continuity of Gaussian measures under certain gauge transformations.
(A.N–, L. Rey-Bellet, S. Sheffield, G. Staffilani).

Bourgain's Method

Let's review Bourgain's framework to prove almost surely GWP and the invariance of a measure **from** LWP.
Consider a dispersive nonlinear Hamiltonian PDE with a k-linear nonlinearity possibly with derivative.

(PDE) $$\begin{cases} u_t = \mathcal{L}u + \mathcal{N}(u) \\ u|_{t=0} = u_0 \end{cases}$$

where $\mathcal{L}$ is a (spatial) differential operator like $i\partial_{xx}$, ∂_{xxx}, etc. (systems). Let $H(u)$ denote the Hamiltonian of (PDE). Then, (PDE) can also be written as

$$u_t = J\,\frac{dH}{du} \quad \text{if } u \text{ is real-valued,} \qquad u_t = J\,\frac{\partial H}{\partial \overline{u}} \quad \text{if } u \text{ is complex-valued.}$$

Let μ denote a measure on the distributions on $\mathbb{T}$, whose invariance we'd like to establish. We assume that μ is a weighted Gaussian measure (formally) given by

$$\text{"}\ d\mu = Z^{-1} e^{-F(u)} \prod_{x\in\mathbb{T}} du(x) \text{"}$$

where $F(u)$ is conserved[3] under the flow of (PDE) and the leading term of $F(u)$ is quadratic and nonnegative.

Now, suppose that there exist a Banach space $\mathcal{B}$ of distributions on $\mathbb{T}$ and a space $X \subset C([-\tau,\tau];\mathcal{B})$ of space-time distributions in which to prove local well-posedness by a fixed point argument with a time of existence τ depending on $\|u_0\|_B$, say $\tau \sim \|u_0\|_B^{-\alpha}$ for some $\alpha > 0$.

[3] $F(u)$ could be the Hamiltonian, but not necessarily!

In addition, suppose that the Dirichlet projections P_N – the projection onto the spatial frequencies $\leq N$ – act boundedly on these spaces, uniformly in N.

Consider the finite dimensional approximation to (PDE)

$$\text{(FDA)} \qquad \begin{cases} u_t^N = \mathcal{L}u^N + P_N(\mathcal{N}(u^N)) \\ u^N|_{t=0} = u_0^N := P_N u_0(x) = \sum_{|n|\leq N} \widehat{u}_0(n)e^{inx}. \end{cases}$$

Then, for $\|u_0\|_B \leq K$ one can see (FDA) is also LWP on $[-\tau, \tau]$ with $\tau \sim K^{-\alpha}$, independent of N.

Two more important assumptions on (FDA):

(1) (FDA) is Hamiltonian with $H(u^N)$ i.e.

$$u_t^N = J\,\frac{dH(u^N)}{d\overline{u}^N}$$

(2) $F(u^N)$ is still conserved under the flow of (FDA)

Note: (1) holds for example when the symplectic form J commutes with the projection P_N. (e.g. $J = i$ or ∂_x.).

In general however (1) and (2) are **not** guaranteed and may not necessarily hold! (more later).

- By Liouville's theorem and (1) above the Lebesgue measure

$$\prod_{|n|\leq N} da_n db_n,$$

where $\widehat{u^N}(n) = a_n + ib_n$, is invariant under the flow of (FDA).

- Then, using (2) - the conservation of $F(u^N)$- we have that the finite dimensional version μ_N of μ:

$$d\mu_N = Z_N^{-1} e^{-F(u^N)} \prod_{|n|\leq N} da_n db_n$$

is also invariant under the flow of (FDA)

- One still needs to prove that μ_N converges weakly to μ (assume it).

- One also needs the following:

Lemma [Fernique-type tail estimate]

For K suff. large, we have

$$\mu_N(\{\|u_0^N\|_{\mathcal{B}} > K\}) < e^{-cK^2}, \text{indep of } N.$$

- This lemma + invariance of μ_N imply the following estimate controlling the growth of solution u^N to (FDA).

Main Proposition: Bourgain '94

Given $T < \infty, \varepsilon > 0$, there exists $\Omega_N \subset \mathcal{B}$ s.t.

- $\mu_N(\Omega_N^c) < \varepsilon$
- for $u_0^N \in \Omega_N$, (FDA) is well-posed on $[-T, T]$ with the growth estimate:

$$\|u^N(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for } |t| \leq T.$$

- Essentially as a corollary of the Main Proposition one can then prove:
  (a) The uniform convergence lemma:

$$\|u - u^N\|_{C([-T,T];\mathcal{B}')} \to 0$$

  as $N \to \infty$ uniformly where $\mathcal{B}' \supset \mathcal{B}$ (for good data).

(b) Given $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \mathcal{B}$ with $\mu(\Omega_\varepsilon^c) < \varepsilon$ such that for $u_0 \in \Omega_\varepsilon$, (PDE) is globally well-posed with the growth estimate:

$$\|u(t)\|_{\mathcal{B}} \lesssim \left(\log \frac{1+|t|}{\varepsilon}\right)^{\frac{1}{2}}, \text{ for all } t \in \mathbb{R}.$$

Note (b) implies that (PDE) is a.s. GWP, since $\widetilde{\Omega} := \bigcup_{\varepsilon>0} \Omega_\varepsilon$ has probability 1.

- Finally, putting all the ingredients together, we obtain the invariance μ: If $\Phi(t)$ is the flow map associated to the nonlinear equation; then for reasonable F

$$\int F(\Phi(t)(\phi))\,\mu(d\phi) = \int F(\phi)\mu(d\phi)$$

Back to the DNLS. Goal 1

- Because the necessary local in time estimates are obtained for the GDNLS, we proceed to construct an invariant weighted Wiener measure for the its flow and prove GWP for data in its support (à la Bourgain).

What's a conserved energy for GDNLS? For v the solution (GDNLS) define

$$\mathcal{E}(v) := \int_{\mathbb{T}} |v_x|^2\,dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v\,v_x}\,dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |v(t)|^2\,dx\right)\left(\int_{\mathbb{T}} |v(t)|^4\,dx\right).$$

$$\mathcal{H}(v) := \mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x - \frac{1}{2}\int_{\mathbb{T}} |v|^4\,dx + 2\pi m(v)^2$$

$$\tilde{\mathcal{E}}(v) := \mathcal{E}(v) + 2m(v)\mathcal{H}(v) - 2\pi\, m(v)^3$$

We prove:

$$\frac{d\tilde{\mathcal{E}}(v)}{dt} = 0.$$

In fact one can show that $E(u) = \tilde{\mathcal{E}}(v)$.

We refer to $\tilde{\mathcal{E}}(v)$ from now on as the *energy* of (GDNLS).

Finite dimensional approximation of (GDNLS)

We consider the dimensional approximation (FGDNLS):

$$v_t^N = iv_{xx}^N - P_N((v^N)^2\overline{v_x^N}) + \frac{i}{2}P_N(|v^N|^4 v^N) - i\psi(v^N)v^N - im(v^N)P_N(|v^N|^2 v^N)$$

with initial data $v_0^N = P_N\, v_0$.

Here,

$$\psi(v^N)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v^N\overline{v}^N{}_x)\,dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v^N|^4 dx - m(v^N)^2$$

and

$$m(v^N)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} |v^N(x,t)|^2 dx.$$

- Note $m(v^N)(t)$ is also conserved under the flow of (FGDNLS).

Grünrock-Herr's LWP estimates for GNLS yield:

Lemma [Local well-posedness]

Let $2 < r < 4$ and $s \geq \frac{1}{2}$. **Then** for every

$$v_0^N \in B_R := \{v_0^N \in \mathcal{F}L^{s,r}(\mathbb{T}) / \|v_0^N\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < K\}$$

and $\tau \sim K^{-\alpha}$, for some $\alpha > 0$, there exists a unique solution

$$v^N \in X^{s,r} \subset C([-\tau, \tau]; \mathcal{F}L^{s,r}(\mathbb{T}))$$

of (FGDNLS) with initial data v_0^N .

Furthermore, by similar arguments as for the 1D NLS, we can prove:

Let $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T}), s \geq \frac{1}{2}, r \in (2,4)$ as in LWP.

Lemma [Approximation lemma]

Assume the solution v^N of (FGDNLS) with initial data $v_0^N(x) = P_N v_0$ satisfies the a priori bound

$$\|v^N(t)\|_{\mathcal{F}L^{s,r}(\mathbb{T})} \leq A, \text{ for all } t \in [-T, T],$$

for some given $T > 0$. **Then** the IVP **(GDNLS)** with initial data v_0 is well-posed on $[-T, T]$ and there exists $C_0, C_1 > 0$, such that its solution $v(t)$ satisfies the following estimate:

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \lesssim \exp[C_0(1+A)^{C_1} T] N^{s_1 - s},$$

for all $t \in [-T, T], 0 < s_1 < s$.

How about the measure?

At this stage we **do not** know that v^N can be globally extended (ie. unlike 1D NLS we do not know the a priori bound above.

Construction of the Weighted Wiener Measure

To construct the measure: use of the conserved quantity $\tilde{\mathcal{E}}(v)$ and the mass.

Hence weighted Wiener rather than Gibbs.

In fact, use the conservation of L^2-norm to slightly modify $\tilde{\mathcal{E}}(v)$ and consider instead the quantity

$$\chi_{\{\|v\|_{L^2}\le B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy $\tilde{\mathcal{E}}(v)$, i.e.

$$\begin{aligned}\mathcal{N}(v) &= -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v}\overline{v}_x\,dx - \frac{1}{4\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\int_{\mathbb{T}}|v|^4\,dx\Big) + \\ &+ \frac{1}{\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\,dx\Big) + \frac{1}{4\pi^2}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)^3.\end{aligned}$$

and B is a (suitably small) constant.

Then we would like to construct the measure (with $v(x) = v_r(x) + iv_i(x)$)

$$\text{``}\, d\mu_\beta = Z^{-1}\chi_{\{\|v\|_{L^2}\le B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx}\prod_{x\in\mathbb{T}} dv_r(x)dv_i(x)\,\text{''}$$

- As before, we have that the associated Gaussian/Wiener measure ρ is a countably additive measure on H^s for any $s < 1/2$ (but **not** for $s \geq 1/2$.). Unfortunately, (GDNLS) is locally well-posed in $H^s(\mathbb{T})$ only for $s \geq \frac{1}{2}$.

- In view of the local well-posedness result by Grünrock-Herr we need to construct ρ as a **measure supported on a Banach space** B. This can be done thanks to the theory of abstract Wiener spaces (Gross and Kuo).

- $B = \mathcal{F}L^{s,r}$ for suitable (s, r).

- Indeed we prove that for $2 \le r < \infty$ and $(s-1)r < -1$:

(1) $(i, H^1, \mathcal{F}L^{s,r})$ is an abstract Wiener space.

B can be realized as the completion of H; $i : H \hookrightarrow B$, inclusion map

(2) The measure ρ can be realized as a countably additive measure supported on $\mathcal{F}L^{s,r}$ and

(3) Have an exponential tail estimate : there exists $c > 0$ (with $c = c(s,r)$) such that

$$\rho(\|v\|_{\mathcal{F}L^{s,r}} > K) \le e^{-cK^2}.$$

- For (r, s) as above $\mathcal{F}L^{s,r}$ scales like H^σ, $\sigma < \frac{1}{2}$.

- We will fix $s = \frac{2}{3}-$ and $r = 3$ and work on this F-L space.

  This pair (s, r) satisfies both the conditions for LWP $s \ge \frac{1}{2}$, $2 < r < 4$ and the conditions for holding the support of the measure $(s-1)r < -1$ above.

More about the construction of the weighted measure

Let

$$R(v) := \chi_{\{\|v\|_{L^2} \le B\}} e^{-\frac{1}{2}\mathcal{N}(v)}, \qquad R_N(v) := R(v^N)$$

where $\mathcal{N}(v)$ is the **nonlinear part of the energy** $\tilde{\mathcal{E}}$.

- We abuse a bit the notation and this of v^N as $P_N(v)$ for some generic function v in our F-L spaces.

After a **nontrivial amount of work** we obtain the weighted Wiener measure:

$$d\mu = Z^{-1} R(v) d\rho,$$

for sufficiently small B, as is the weak limit of the finite dimensional weighted Wiener measures μ_N on $\mathbb{R}^{4N+2}$ given by

$$\begin{aligned} d\mu_N &= Z_N^{-1} R_N(v) d\rho_N \\ &= \hat{Z}_N^{-1} \chi_{\{\|\widehat{v}^N\|_{L^2} \le B\}} e^{-\frac{1}{2}(\mathcal{E}(\widehat{v}^N) + \|\widehat{v}^N\|_{L^2})} \prod_{|n| \le N} da_n db_n \end{aligned}$$

for suitable normalizations Z_N, $\hat{Z}_N$. More precisely we have:

Lemma [Convergence]

$R_N(v)$ converges in measure to $R(v)$.

The proof is probabilistic. Need to rely on Standard Deviation type estimates.

All in all we have:

Proposition [Existence of weighted Wiener measure]

(a) For sufficiently small $B > 0$, we have $R(v) \in L^2(d\rho)$. In particular, the weighted Wiener measure μ is a probability measure, *absolutely continuous* with respect to the Wiener measure ρ.

(b) We have the following tail estimate. Let $2 \leq r < \infty$ and $(s-1)r < -1$; then there exists a constant c such that

$$\mu(\|v\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2}$$

for sufficiently large $K > 0$.

(c) The finite dim. weighted Wiener measure μ_N converges weakly to μ.

a.s GWP : analysis of the (FGDNLS)

Next, the key step is to prove the analogue of Bourgain's Main Proposition above controlling the growth of solutions v^N to (FGDNLS).

Is μ_N invariant ? ...

Obstacles we have to face:

- The symplectic form associated to the periodic gauged derivative nonlinear Schrödinger equation GDNLS does not commute with Fourier modes truncation and so the truncated finite-dimensional systems are not necessarily Hamiltonian. This entails two problems:
  - (1) A mild one: need to show the invariance of Lebesgue measure associated to (FGDNLS) ('Liouville's theorem') by hand directly .
  - (2) A more serious one and at the heart of this work. The energy $\tilde{\mathcal{E}}(v^N)$ is **no** longer conserved. In other words, the finite dimensional weighted Wiener measure μ_N is **NOT** invariant any longer[4] .

[4] Zhidkov faced a similar problem but unlike his work on KdV, **we do not have** a priori knowledge of global well posedness.

Almost conserved energy

- We prove that μ_N is *almost* invariant in the sense that we can control the growth in time of the energy $\tilde{\mathcal{E}}$ of the solution v^N to the *finite dimensional approximation equation*.
- More precisely, we have the following estimate controlling the growth of $\tilde{\mathcal{E}}(v^N)(t)$

Theorem [Energy Growth Estimate]

Let $v^N(t)$ be a solution to (FGDNLS) in $[-\tau, \tau]$, and let $K > 0$ be such that $\|v^N\|_{X^{\frac{2}{3}-,3}} \leq K$. Then there exists $\beta > 0$ such that

$$|\tilde{\mathcal{E}}(v^N(\tau)) - \tilde{\mathcal{E}}(v^N(0))| = \left| \int_0^\tau \frac{d}{dt} \tilde{\mathcal{E}}(v^N)(t) dt \right| \lesssim C(\tau) N^{-\beta} \max(K^6, K^8).$$

$$
\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{E}}(v^N) = & -2\mathrm{Im}\int v^N \overline{v^N v_x^N} P_N^{\perp}((v^N)^2 \overline{v_x^N}) + \mathrm{Re}\int v^N \overline{v^N v_x^N} P_N^{\perp}(|v^N|^4 v^N) \\
& - 2m(v^N)\mathrm{Re}\int v^N \overline{v^N v_x^N} P_N^{\perp}(|v^N|^2 v^N) \\
& + 2m(v^N)\mathrm{Re}\int v^N \overline{v^N}^2 P_N^{\perp}((v^N)^2 \overline{v_x^N}) \\
& + m(v^N)\mathrm{Im}\int v^N \overline{v^N}^2 P_N^{\perp}(|v^N|^4 v^N) \\
& - 2m(v^N)^2\mathrm{Im}\int v^N \overline{v^N}^2 P_N^{\perp}(|v^N|^2 v^N) + \ldots\ldots,
\end{aligned}
$$

The first term is the worst term since it has two derivatives.

Take away: We now have that μ_N is *almost invariant*.

Growth of solutions to (FGDNLS)

Armed with the Energy Growth Estimate we count on the almost invariance of the finite-dimensional measure μ_N under the flow of (FGDNLS) to control the growth of its solutions (our analogue of Bourgain's Main Proposition)

Proposition [Growth of solutions to FGDNLS]

For any given $T > 0$ and $\varepsilon > 0$ and N large there exist sets $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T)$ in $\mathcal{F}L^{\frac{2}{3}-,3}$ such that:

(a) $\mu_N\left(\widetilde{\Omega}_N\right) \geq 1 - \varepsilon$.

(b) For any initial condition $v_0^N \in \widetilde{\Omega}_N$, (FGDNLS) is well-posed on $[-T, T]$ and its solution $v^N(t)$ satisfies the bound

$$\sup_{|t|\leq T} \|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

A.S GWP of solution to (GDNLS)

Combining the Approximation Lemma of v by v^N with the previous Proposition on the growth of solutions to (FGDNLS) we can prove a similar result for solutions v to (GDNLS):

Proposition [‘Almost almost ’ sure GWP for (GDNLS)]

For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon, T)$ such that

(a) $\mu\left(\Omega(\varepsilon, T)\right) \geq 1 - \varepsilon$.

(b) For any initial condition $v_0 \in \Omega(\varepsilon, T)$ the IVP (GDNLS) is well-posed on $[-T, T]$ with the bound

$$\sup_{|t| \leq T} \|v(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

All in all we now have:

Theorem 1 [Almost sure global well-posedness of (GDNLS)]

There exists a set Ω, $\mu(\Omega^c) = 0$ such that for every $v_0 \in \Omega$ the IVP (GDNLS) with initial data v_0 is globally well-posed.

Theorem 2 [Invariance of μ]

The measure μ is invariant under the flow $\Phi(t)$ of (GDNLS)

Finally: The last step is going back to the ungauged (DNLS) equation. By pulling back the gauge, it follows easily from Theorems 1 and 2 that we have:

The ungauged DNLS equation

Recall, μ is a measure on Ω and $\mathcal{G}^{-1} : \Omega \to \Omega$ measurable. Define the measure $\nu = \mu \circ \mathcal{G}$ by

$$\nu(A) := \mu(\mathcal{G}(A)) = \mu(\{v : \mathcal{G}^{-1}(v) \in A\}).$$

for all measurable sets A or equivalently - for integrable F- by

$$\int F d\nu = \int F \circ \varphi \, d\mu$$

Theorem 3 [Almost sure global well-posedness of (DNLS)]

There exists a subset Σ of the space $\mathcal{F}L^{\frac{2}{3}-,3}$ with $\nu(\Sigma^c) = 0$ such that for every $u_0 \in \Sigma$ the IVP (DNLS) with initial data u_0 is globally well-posed.

Finally we show that the measure ν is invariant under the flow map of DNLS.

Theorem 4 [Invariance of measure under (DNLS) flow]

The measure $\nu = \mu \circ \mathcal{G}$ is invariant under the (DNLS) flow.

Goal 2

What is $\nu = \mu \circ \mathcal{G}$ really? Is it absolutely continuous with respect to the measure that can be naturally constructed for DNLS by using its energy E,

$$\begin{aligned} E(u) &= \int_{\mathbb{T}} |u_x|^2\,dx + \frac{3}{2}\mathrm{Im}\int_{\mathbb{T}} u^2\overline{uu_x}\,dx + \frac{1}{2}\int_{\mathbb{T}} |u|^6\,dx \\ &=: \int_{\mathbb{T}} |u_x|^2\,dx + \mathcal{K}(u) \end{aligned}$$

as done by Thomann-Tzevtkov?

We know ν is invariant and that the ungauged (DNLS) equation is GWP a.s with respect to ν. Treating the weight is easy. The problem is ungauging the Gaussian measure ρ.

Question: What is $\tilde{\rho} := \rho \circ \mathcal{G}$? Is (its restriction to a sufficiently small ball in L^2) absolutely continuous with respect to ρ? If so, what is its Radon-Nikodym derivative?

We would like to compute $\tilde{\rho}$ explicitly. This turns out to be an intricate problem that requires tools from stochastic analysis and probability.

Absolute continuity of Brownian bridges under gauge transformations

- There is an analytic theory on Gaussian measures under nonlinear transformations both of non-anticipative and anticipative type (see eg. Bogachev's book and references therein).

- This theory is fairly well understood for transformations of the form $x + F(x)$ with F a transformation from a Banach space (associated to the support of the measure) into a Hilbert space H, known as the Cameron-Martin space (associate to the construction of measure; $H = \dot{H}^1$ in the case of DNLS above) and whose (Fréchet) derivative F' in the direction of H exists and is 'nice', for example $F'|_H$ is Hilbert-Schmidt.

- **But** this framework does not fit (directly) the gauge transformations as the one above. Gauge transformations as the ones used above are L^2 unitary transformations which do not have this $I + F$ form.

The work of Cambronero-McKean on the periodic KdV and the Miura gauge transformation- exploited by Quastel and Valkó- is not directly applicable either.

The ungauged measure: absolute continuity

In order to finish this step one should stop thinking about the solution v as a infinite dimension vector of Fourier modes and start thinking instead about v as a (periodic) complex Brownian path in $\mathbb{T}$ (Brownian bridge) solving a certain stochastic process.

We recall that to ungauge we need to define

$$\mathcal{G}^{-1}(v)(x) := \exp(iJ(v))\, v(x)$$

where

$$J(v)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x |v(y)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2(\mathbb{T})}\, dy\, d\theta$$

It will be important later that $J(v)(x) = J(|v|)(x)$. Then, if v satisfies

$$dv(x) = \underbrace{dB(x)}_{\text{Brownian motion}} + \underbrace{b(x)dx}_{\text{drift terms}}$$

by **Ito's calculus** and since $\exp(iJ(v))$ is differentiable we have:

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\, dv + iv\exp(iJ(v))\left(|v(x)|^2 - \frac{1}{2\pi}\|v\|^2_{L^2}\right) dx + \dots$$

What one may think it saves the day...

Substituting above one has

$$d\mathcal{G}^{-1}v(x) = \exp(iJ(v))\,[dB(x) + a(v,x,\omega))\,dx] + \dots$$

where

$$a(v,x,\omega) = iv\ \left(|v(x)|^2 - \frac{1}{2\pi}\|v\|_{L^2}^2\right).$$

What could help?

- The fact that $\exp(iJ(v))$ is a unitary operator
- The fact that one can prove Novikov's condition:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int a^2(v,x,\omega)dx\right)\right] < \infty.$$

In fact this last condition looks exactly like what we'd need for the following:

'Theorem' (Girsanov)

If we change the drift coefficient of a given Ito process in an appropriate way, then the law of the process will not change dramatically. In fact the new process law will be absolutely continuous with respect to the law of the original process and we can compute explicitly the Radon-Nikodym derivative.

Why Girsanov's theorem doesn't save the day

However, if one reads the theorem carefully one realizes that an important condition is that $a(v, x, \omega)$ is *non anticipative*; in the sense that it only depends on the BM $v(x)$ up to "time" x and not further. This unfortunately is not true in our case! The new drift term $a(v, x, \omega)$ involves the L^2 norm of $v(x)$ (periodic case!) and hence it is *anticipative*. A different strategy is needed ...

Conformal invariance of complex BM comes to the rescue!

We use the well known fact that if $W(t) = W_1(t) + iW_2(t)$ is a complex Brownian motion, and ϕ is an analytic function then $Z = \phi(W)$ is, after a suitable time change, again a complex Brownian motion.

In what follows think of $Z(t)$ to play the role of our complex BM $v(x)$

Next take the ϕ to be the exponential.

For $Z(t) = \exp(W(s))$ the time change is given by

$$t = t(s) = \int_0^s |e^{W(r)}|^2 dr, \qquad \frac{dt}{ds} = |e^{W(s)}|^2,$$

equivalently

$$s(t) = \int_0^t \frac{dr}{|Z(r)|^2}, \qquad \frac{ds}{dt} = \frac{1}{|Z(t)|^2}.$$

We are interested in $Z(t)$ for the interval $0 \leq t \leq 2\pi$ and thus we introduce the stopping time

$$\mathcal{S} = \inf \left\{ s\,;\ \int_0^s |e^{W(r)}|^2 dr \ = \ 2\pi \right\}$$

Important: The stopping time $\mathcal{S}$ depends only on the **real part** $W_1(s)$ of $W(s)$ (or equivalently only $|Z|$).

If we write $Z(t)$ in polar coordinate $Z(t) = |Z(t)|e^{i\Theta(t)}$ we have

$$W(s) = W_1(s) + iW_2(s) = \log|Z(t(s)| + i\Theta(t(s))$$

and W_1 and W_2 are real independent Brownian motions.

If we define

$$\tilde{W}(s) := W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(|Z|)(r)dr\right]$$
$$= W_1(s) + i\left[W_2(s) + \int_0^{t(s)} h(e^{W_1})(r)dr.\right]$$

In our case, essentially $h(|Z|)(\cdot) = |Z(\cdot)|^2 - \|Z\|_{L^2}^2.$

We then have

$$e^{\tilde{W}(s)} = \tilde{Z}(t(s)) = \mathcal{G}^{-1}(Z)(t(s)).$$

- In terms of W, the gauge transformation is now easy to understand. It gives a complex process such that:
  - The real part is left unchanged.
  - The imaginary part is translated by the function $J(Z)(t(s))$ which depends only on the real part (ie. on $|Z|$, which has been fixed) and in that sense is deterministic.
  - It is now possible to use Cameron-Martin-Girsanov's theorem only for the law of the imaginary part and conclude:

Conclusion

Then if η denotes the probability distribution of W and $\tilde{\eta}$ the distribution of $\tilde{W}$ we have the absolute continuity of $\tilde{\eta}$ and η whence the absolute continuity between $\tilde{\rho}$ and ρ follows with the **same Radon-Nikodym derivative** (re-expressed back in terms of t).

All in all then we prove that our ungauged measure ν is in fact essentially (up to normalizing constants) of the form

$$d\nu(u) = \chi_{\|u\|_{L^2} \leq B} e^{-\mathcal{K}(u)} d\rho,$$

the weighted Wiener measure associated to DNLS (constructed by Thomann-Tzvetkov). **In particular we prove its invariance.**

- The above needs to be done carefully for complex Brownian bridges (periodic BM) by conditioning properly.
  - $W(s)$ is a BM conditioned to end up at the same place when the total variation time $t = t(s)$ reaches 2π. The time when this occurs is our $\mathcal{S}$.
  - Conditioned on ReW we have that ImW is just a regular real-valued BM conditioned to end at the same place (up to multiple of 2π) where it started at time $\mathcal{S}$