The large box limit of nonlinear Schrodinger equations in weakly nonlinear regime : ①

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(Joint w/ Buckmaster, Germain, Hani)

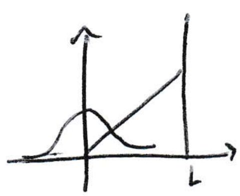
$$i u_t = \frac{1}{2\pi} \Delta u + |u|^{p-1} u \qquad p \text{ odd integer}, \quad x \in [0,L]^D$$

$$u(0,x) = \varepsilon u_0(x)$$

By scaling $\Longrightarrow$
$$i u_t = \frac{1}{2\pi} \Delta u + \varepsilon^{p-1} |u|^{p-1} u$$
$$u(0,x) = u_0(x)$$

Long-time behavior ? I.e. when $T > \frac{1}{\varepsilon^{p-1}}$ (to see nonlinear effects)

when $T > L$ (to see the boundary)



$$u(x,t) = \frac{1}{L^D} \sum_k{}' a_k \, e\left(\frac{kx}{L}\right) e\left(\frac{k^2 t}{L^2}\right) \quad \text{where } e(z) = e^{2\pi i z}$$

$$u(x,0) = \frac{1}{L^D} \sum a_{0,k} \, e\left(\frac{kx}{L}\right), \quad a_0(k) = g\left(\frac{k}{L}\right).$$

$$i\partial_t a_k = \frac{\varepsilon^{p-1}}{L^{D(p-1)}} \sum_{S(k)=0}{}' a_{k_1} \overline{a_{k_2}} \cdots a_{k_p} \, e\left(\frac{\Omega t}{L^2}\right)$$

$$S(k) = k_1 - k_2 + \cdots + k_p - k, \quad \Omega(k) = k_1^2 - k_2^2 + k_3^2 - \cdots + k_p^2 - k^2$$

"THM":
$$i\partial_t g = \varepsilon^{p-1} Z(L)\, T(g) + \varepsilon^{p-1} Z_1(L)\, A(g) + \text{error}$$

$$Z(L) \sim \frac{1}{L^{(p-1)D-2}}, \quad t = \frac{1}{\varepsilon^{p-1} Z(L)} \sim \frac{L^{(p-1)D-2}}{\varepsilon^{p-1}}$$

$$\sup_k (1+|k|^2)^6 \left| a_k - g\left(\frac{k}{L}, t\right) \right| \ll \delta$$

Resonances: $$i\partial_t a_k = \frac{\varepsilon^{p-1}}{L^{p(p-1)}} \sum_{\substack{S(k)=0 \\ \Omega(k)=0}} a_{k_1} \cdots a_{k_p}.$$

$$\sum_{F(x)=0} W\left(\frac{x}{L}\right)$$

set should contain L^{n-2} points, since F is quadratic.

$$\int_0^1 \sum W\left(\frac{x}{c}\right) e(\alpha F(x))\, dx$$

circle method should work if

$$L^{n-2} \qquad n-2 > \frac{n}{2} \qquad L^{n/2}.$$

$$\Rightarrow \quad D(p-1) - 4 > \frac{D(p-1)-2}{2}$$

$D=1$, $p>5$ $\qquad$ $D \geq 3$, $p \geq 3$.

$D=2$, $p>3$ $\frac{L^2}{\ln L}$ $\qquad$ F.G.H.

$$\int_0^1 \sum_x W\left(\frac{x}{L}\right) e(\alpha F(x))\, d\alpha = \sum_{c,q} S_q(c)\, I_q(c)$$

use a partition of unity to separate rational numbers of large vs. small denominators.

$$I_q(0) \longrightarrow \int W(z)\, \delta(F(z))\, dz$$

When $n-2 > \frac{n}{2}$, $\quad S_q(0) \longrightarrow L^{n-2}$, $\quad O(L^{n-\frac{5}{2}})$.

When $n-2 = \frac{n}{2}$ $\quad S_q(0) \longrightarrow L^2 \ln L, \qquad L^2 A(\omega)$

Rather, $\quad L^2 \ln L \int W(z)\, \delta(F)\, dz = 0 + L^2 A(\omega) + O(L^{\frac{3}{2}})$.

$\underline{D = 2,\ p = 3}$

$$i \partial_t g = \frac{2 \varepsilon^2}{\zeta(2) L^2} \ln(L) \int g_1 \bar{g}_2 g_3\, \delta(S)\, \delta(\Omega)\, dk_1\, dk_2 + L^2 A(g)$$

$$\sup_k (1 + |k|^2)^6 \left| a_k(t) - g\left(\tfrac{k}{L}, t\right) \right| \lesssim \frac{\varepsilon^2}{L^{\frac{1}{2}}} \qquad t \sim \frac{L^2}{\varepsilon^2 \ln L}.$$

$$\sum_{S(k)}' (a_1 \bar{a}_2 a_3)\, e\left(\frac{\Omega_3 t}{L^2}\right) = \sum_{\substack{S(k)=0 \\ \Omega=0}}' a_1 \bar{a}_2 a_3 + \sum_{\substack{S=0 \\ \Omega \neq 0}} a_1 \bar{a}_2 a_3\, e\left(\frac{\Omega_3 t}{L^2}\right)$$

$$\frac{d}{dt}\left[\sum' \frac{L^2}{\Omega_3} a_1 \bar{a}_2 a_3 \cdot e\left(\frac{\Omega_3 t}{L^2}\right) \right] - \underbrace{\sum' \frac{L^2}{\Omega_3} \dot{a}_1 \bar{a}_2 a_3\, e\left(\frac{\Omega_3 t}{L^2}\right)}_{-\sum' \frac{1}{\lambda} \sum'_{\Omega_3 = \lambda}}$$

$\dot{a}_1 = O(\varepsilon^2)$.

first normal form $\quad \varepsilon^2 L^2$ $\longleftarrow$ need small

second normal form $\quad \varepsilon^4 \cdot L^2$

" ——— " $\quad \varepsilon^{2k} \cdot L^{2+} \implies \varepsilon < \frac{1}{L^\delta}$