10/30 Pierre Germain

Long wave limits for schrödinger maps.

Water waves

KdV $\qquad \partial_t u + \partial_x^3 u + u \partial_x u = 0$

$$u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Bou 1870

Korteweg - de Vries 1895

Craig 1985 Alvazerg - Samarego Lannes

Another context NLS with potential

$$i \partial_t \psi - \partial_x^2 \psi = (|\psi^2| - 1) \psi$$

Gross - Pitaevski $\qquad \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$

physics. Kiushai - Anderson - Lisae $\qquad$ Kuznetsov - Turitsyn

Math. Bethued - Gravegat - Saut - Smels

Chiron - Rouset $\qquad$ Chiron

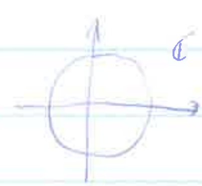
Bunier $\qquad$ Lin - Zheng

Many other context where KdV is derived in a long-wave limit.

Vlasov - Poisson $\qquad$ Euler - Poisson.

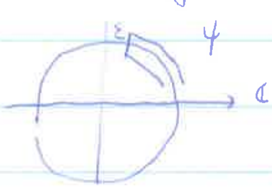
The case of Gross - Pitaevski

$$i \partial_t \psi - \partial_x^2 \psi = -(|\psi|^2 - 1) \psi$$



Energy $\int (|\partial_x \psi|^2 + (|\psi|^2 - 1)^2) \, dx$

Wave scaling



time scale (t) $\frac{1}{\varepsilon}$

space scale (x) $\frac{1}{\varepsilon}$

Modeling transform

$\psi = \sqrt{\rho}\, e^{i\varphi}$

ψ solves GP $\Leftrightarrow$ $(\rho, \nabla\varphi)$ compressible fluid.

$\psi = \sqrt{1 + \varepsilon a(\underbrace{\varepsilon t}_{T}, \underbrace{\varepsilon x}_{X})}\; e^{i\varphi(\varepsilon t, \varepsilon x)}$ $\qquad u = \partial_X \varphi$

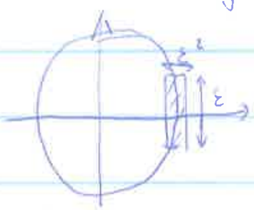
(GP)

$$\Leftrightarrow \quad \partial_T a + \partial_X u = -\varepsilon\, \partial_X(au)$$

$$\partial_T u + \partial_X a = -\varepsilon(u\,\partial_X u) + \varepsilon\, \partial_X\left(\frac{\partial_X^2 \sqrt{\rho}}{\rho}\right)$$

As $\varepsilon \to 0$ $\quad \begin{cases} \partial_T a + \partial_X u = 0 \\ \partial_T u + \partial_X a = 0 \end{cases} \qquad (\partial_T^2 - \partial_X^2)\, a = 0$

KdV scaling.



time scale $\sim \frac{1}{\varepsilon^3}$

space scale $\sim \frac{1}{\varepsilon}$

Modeling transform

$$\psi = [1 + \varepsilon^2 a(\varepsilon(x-t), \varepsilon^3 t)]\, e^{i\varepsilon\varphi(\varepsilon(x-t), \varepsilon^3 t)}$$

$\to$ as $\varepsilon \to 0$, a solves KdV.

$$\partial_t a + \partial_x^3 a + a\,\partial_x a = 0$$

A general framework

$\psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. $\qquad u : \mathbb{R} \times \mathbb{R} \to M$. Kähler manifold.

M: complex structure $i : T_p M$ ↺

isometry, $[i, \nabla] = 0$, $i^2 = -1$

Schrödinger $\quad \underbrace{i\,\partial_t u}_{\in T_u M} + \nabla_x \partial_x u = V'(u) \quad V$ potential on M.

Energy $\int (|\partial_x u|^2 + V(u))\, dx$

- V is minimal on $\{V = 0\} = \mathcal{L}$
- $\mathcal{L}$ is Lagrangian $\quad i$ exchanges $T_u\mathcal{L}$ and $N_u\mathcal{L}$

① Gross - Pitaevski

$u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$

V minimal on $\{ |u| = 1 \} = \mathcal{L}$ | complex structure i.

②. Vector valued G-P

$u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$

③ Landau - Lifshitz for ferromagnetic chains.

$u : \mathbb{R} \times \mathbb{R} \to S^2$

complex structure $\quad i = u \times$.

$\partial_x u = u \times (\partial_x^2 u - V'(u))$

$\{ V = 0 \} = \mathcal{L}$

S^2.

④. Landau - Lifshitz for anti ferromagnetic chains

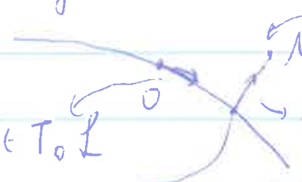
$u : \mathbb{R} \times \mathbb{R} \longrightarrow S^2 \times S^2$.

$\partial_t u = u \times (- \partial_{xx} u + v - \partial_x v)$

$\partial_t v = v \times (- \partial_{xx} v + u + \partial_x u)$

Potential $\int |u + v|^2$ minimal on $\{ u = - v \}$ $\quad S^2 \times S^2$

Taking coordinates.



M $\quad u = \exp_p^M (\underline{n}) = \psi (p, n)$

$n \in N_p \mathcal{L}$

$x \in T_0 \mathcal{L}$ $\quad 0$

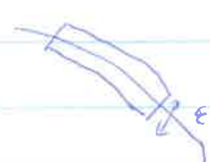
$\mathcal{L} = \{ V = 0 \}$

$p = \exp_0^{\mathcal{L}} (x)$

Thm (wave scaling) [Shatah - Zeng].

Consider solutions of the type

$u(t, x) = \psi (p (\varepsilon t, \varepsilon x) , \varepsilon n (\varepsilon t, \varepsilon x))$.



ε

$\mathcal{L}$

As $\varepsilon \to 0$ dynamics given by.

time scale $\sim \frac{1}{\varepsilon}$

space scale $\sim \frac{1}{\varepsilon}$

wave maps eq. on $\mathcal{L}$

(WM) on $\mathcal{L}$ $\quad \nabla_t^{\mathcal{L}} \partial_t p - \nabla_x^{\mathcal{L}} \partial_x p = 0$

Thm. (KdV scaling) [G. – Frédéric Rousset (Orsay)]

time scale $\sim \frac{1}{\varepsilon^3}$

space scale $\sim \frac{1}{\varepsilon}$

$$u(t,x) = \underset{\mathcal{M}}{\Psi}\Big(\underset{\mathcal{L}}{\phi}\big(\varepsilon\varphi(\underbrace{\varepsilon^3 t}_{T}, \underbrace{\varepsilon(x-t)}_{X})\big), \varepsilon^2 m(\underbrace{\varepsilon^3 t}_{T}, \underbrace{\varepsilon(x-t)}_{X})\Big)$$

As $\varepsilon \to 0$ dynamics are given by $\qquad \varphi \in \mathbb{R}^d$

$$2\partial_t \varphi = \frac{1}{4}\partial_x^3 \varphi + B(\varphi, \partial_x \varphi) \qquad B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d.$$

depends on V on second fund. form of $\mathcal{L}$.

<u>Questions</u>

* $V \sim c|n|^2$ what if this is not the case.
* $\partial_t u + \partial_x^3 u + B(u, \partial_x u) = 0$

  $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$. $\qquad B. \ \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$

→ LWP. $H^{-\frac{3}{4}+}$ Kenig - Ponce – Vega.

→ Traveling waves!

→ complete integrability?

→ asymptotic behavior?