HYPERBOLIC-LIKE BEHAVIOUR OF GROUPS

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ABSTRACT. I will discuss properties, techniques and examples related to hyperbolic-like groups. For example, contracting geodesics, weakly proper discontinuous/acylindrical group actions. Then I explain the construction of projections complexes and mention some of its applications.

1.

A geodesic space X is *δ-hyperbolic* if every geodesic triangle is δ-thin.

A group G is *word-hyperbolic* if $\exists\, X$ δ-hyperbolic such that $G \curvearrowright X$ by isometries, properly, co-boundedly. Properly: for fixed $x \in X$, for all $R > 0$, $\#\{g \in G \mid d(x, gx) < R\} < \infty$. Co-boundedly: $\exists\, R$, $G \cdot B(x, R) = X$.

(Throughout talk, all actions are by isometries.)

Theorem (/definition)**.** *G is word-hyperbolic $\iff$ G is f.g. and* $\mathrm{Cayley}(G)$ *is δ-hyperbolic.*

$\Leftarrow$ is straightforward.

$\Rightarrow$ by first applying Svarc–Milnor to get G f.g. and $\mathrm{Cayley}(G) \sim_{QI} X$ δ-hyperbolic (quasi-isometric). Then we are finished, as hyperbolicity is a QI-invariant.

Key step in proof that hyperbolicity is a QI-invariant is

Lemma (Morse lemma)**.** *(K, L)-quasi-geodesic in a δ-hyperbolic space X, then it is a bounded distance ($\leq C(K, L, \delta)$) from a geodesic.*

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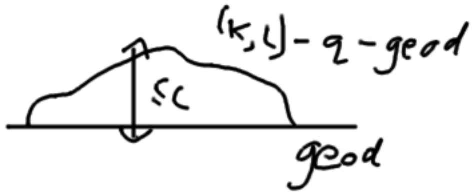


2. RANK-1 GEOD

Example/non-examples of hyperbolic spaces.

- Trees are 0-hyperbolic, free groups are word-hyperbolic.
- $\mathbb{H}^2$ is hyperbolic, $\pi_1(\Sigma_g), g \geq 2$ is word-hyperbolic.
- $\mathbb{E}^2$ is not hyperbolic, $\mathbb{Z}^2$ is not word-hyperbolic.
- M a closed Riemannian manifold of sectional curvature $K \leq 0$. $G = \pi_1 M \curvearrowright \widetilde{M}$ by isometries, properly, co-compactly. $\widetilde{M}$ is an "Hadamard" manifold, CAT(0) space, but maybe not δ-hyperbolic.

Rank-rigidity theorem (Ballmann): M is either

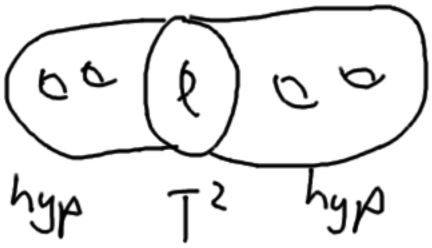
(1) a product $M_1 \times M_2$, $G = G_1 \times G_2$, $|G_i| = \infty$, G is not word-hyperbolic.

(2) a locally symmetric space of $rk \geq 2$, $\widetilde{M} > \mathbb{E}^2$, G is not word-hyperbolic.

(3) a "rank-1 manifold": e.g.

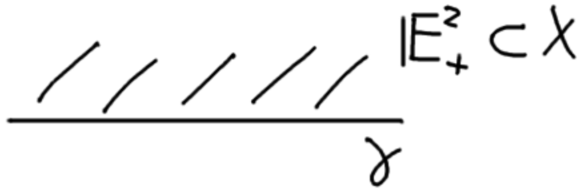
   (a) M is hyperbolic, G is word-hyperbolic

   (b) 3-dimensional manifold, two hyperbolic manifolds glued along a torus cusp:



G is not word-hyperbolic, $\pi_1 T^2 = \mathbb{Z}^2 < G$.

- In Hadamard manifold / CAT(0) space X, an ∞-geodesic γ is *rank-1* if it does not bound a flat half plane.

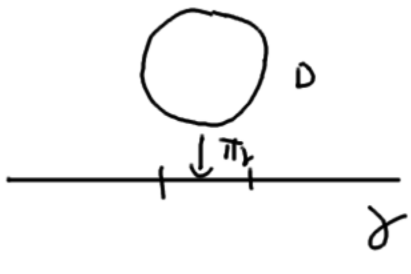


- A hyperbolic space X with a geodesic axis γ is *rank-1* if γ is rank-1.
- A manifold M of $K \leq 0$ is *rank-1* if $\exists\, g \in \pi_1 M$ that is rank-1 on $\widetilde{M}$.

For geometric group theorists, consider case (3) in Rank-rigidity Theorem to be the general case.

3. CONTRACTING GEODESICS

Let $\gamma \subset X$ a geodesic space, γ a (quasi-)geodesic. Let $B > 0$. We say γ is *(B-)contracting* if for every metric ball $D \subset X$ such that $D \cap \gamma = \emptyset$, we have diam $\pi_\gamma \leq B$, where $\pi_\gamma : X \to \gamma$ is nearest point projection.



Lemma. *M a Riemannian manifold, $K \leq 0$, $1 \neq g \in \pi_1 M$ hyperbolic with axis γ. Then g, or equivalently γ, is rank-1 $\iff$ $\exists\, B, \gamma$ is B-contracting.*

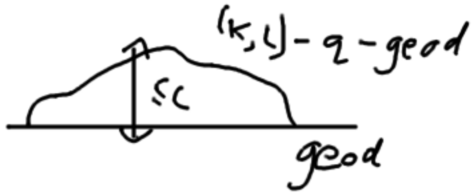
$\Leftarrow$ is trivial, $\Rightarrow$ need some work.

Exercise. If X is δ-hyperbolic, every geodesic is 10δ-contracting.

Theorem (Minsky). *Every pseudo-Anosov ("pA") in* MCG(Σ) *has a B-contracting geodesic axis in* Teich(Σ).

Reminder. Teich(Σ) is not δ-hyperbolic.

Morse lemma. a B-contracting geodesic γ satisfies Morse lemma:



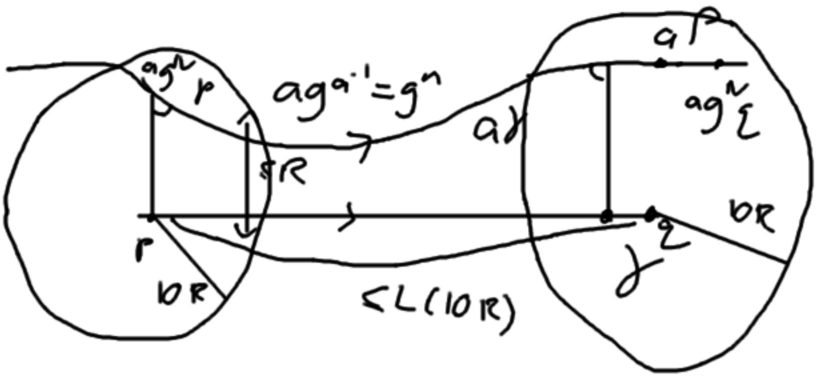
where now $C = C(K, L, B)$.

Sample proposition. $G \curvearrowright X$ properly, $\exists g \in G$ hyperbolic with a quasi-geodesic axis γ, B-contracting. Then normalizer $N_G(\langle g \rangle)$ is virtually $\mathbb{Z}$.

Remark. Applies to all pA in MCG.

Suppose $a \in N_G(\langle g \rangle)$.

G is word-hyperbolic if $G \curvearrowright X$, δ-hyperbolic properly co-boundedly.



4. WPD

MCG(Σ) $\curvearrowright C(\Sigma)$, the curve complex, which is δ-hyperbolic. Every pA $\in$ MCG is hyperbolic with a quasi-geodesic axis γ.

But $C(\Sigma)$ is not proper, the action is not proper.

Definition. $G \curvearrowright X$ geodesic space, $g \in G$ hyperbolic with axis γ. We say g is *weakly properly discontinuous* (WPD) if $\forall\, R > 0, \exists\, L$ such that $\forall\, x, y \in \gamma$ satisfying $|x - y| > L$

$$\#\,\{a \in G\ :\ |x - ax| \leq R \text{ and } |y - ay| \leq R\} < \infty.$$

Remark. $G \curvearrowright X$ is proper $\implies$ every $g \in G$ is WPD.

Proposition. *Every* $pA \in \mathrm{MCG}(\Sigma)$ *is WPD on* $C(\Sigma)$.

Sample proposition. $G \curvearrowright X$, $\exists\, g \in G$, hyperbolic, WPD with a B-contracting axis γ. Then $N_G(\langle g \rangle)$ is virtually $\mathbb{Z}$.

This demonstrates the advantage of WPD that we can consider properness of a single element (while the whole action is not proper).

Remark. Applies to a pA $\in$ MCG $\curvearrowright C(\Sigma)$.

Summary theorem. (Bestvina–Bromberg–F.) If G acts on X such that $\exists\, g \in G$, hyperbolic and WPD with a B-contracting axis γ, then G acts on some quasi-tree Q by isometries, such that $g \in G$ is hyperbolic and WPD. (Quasi-tree Q: a geodesic space Q quasi-isometric to some simplicial tree, δ-hyperbolic, e.g. Farey graph.)

Example.

- Discrete subgroups in Isom $\mathbb{H}^n$, $\forall\, g \in G$ hyperbolic element.
- G hyperbolic space, $\forall\, g$ of ∞-order.
- MCG, $\forall$ pA $\curvearrowright C(\Sigma)$.
- $\mathrm{Out}(F_n)$, $\forall$ fully irreducible $\curvearrowright$ Outer space.
- π_1 of rank-1 manifold, for every rank-1 element.

Non-example. $\mathrm{SL}_3\,\mathbb{Z}, \forall\, g$

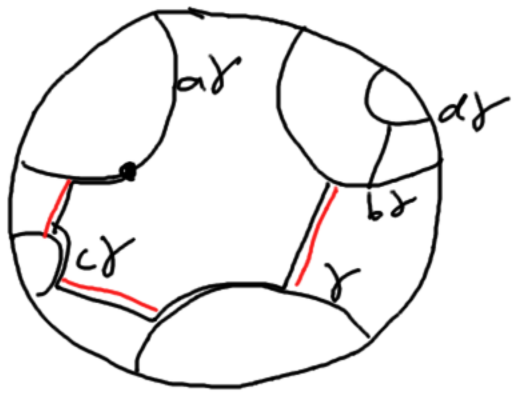
Sample application.

Theorem (Dahmani–Guirardel–Osin). *If $G \curvearrowright X$ δ-hyperbolic, $\exists\, g \in G$ hyperbolic and WPD, $\exists\, N$ large such that gN normally generates a free subgroup of rank ≥ 2 in G, unless G is virtually $\mathbb{Z}$.*

5. Projection complex

Setting. $G \curvearrowright X, \exists\, g$ hyperbolic, WPD, B-contracting γ.

$g \in \pi_1\Sigma \curvearrowright \mathbb{H}^2$



$Y = \{a\gamma \mid a \in G\} / \sim$

Lemma. *$\exists\, L$ such that $\forall\, a \in G$ either $\gamma \sim a\gamma$ (Hausdorff distance $Hd(\gamma, a\gamma) \leq L$) or* diam *$\pi_\gamma(a\gamma) \leq L$.*

$V(Q) = Y$. There is a rule to join two points in Y.

Called projection complex.

axiom

Sample theorem. $\exists\, \Gamma <$ MCG finite index, $\forall\, g \in \Gamma$ ∞-order $\implies$ $\exists\, \Gamma \curvearrowright P$ δ-hyperbolic such that g is hyperbolic (not WPD, WWPD).

In particular, g is not distorted in G, $\|g^n\|$ grows linearly.

Theorem (Farb–Lubotzky–Minsky). *Every $g \in$ MCG of ∞-order is not distorted.*

6. PROBLEMS

Prove:	Hyp. group	MCG	Out(F_n)
Assume f.g. G is "hyperbolic-like": $\forall g \in G$, ∞-order, maybe passing to a finite index subgroup of G, $G \curvearrowright X$, hyperbolic space such that g is hyperbolic, WPD / WWPD (weakly WPD).	yes	yes (BBF)	?
Then G has no distortion.	yes	yes (FLM)	yes
G satisfies a quadratic isoperimetric inequality.	yes	yes	no (exponential)
G acts on some l^p-space, isom, proper.	yes (Yu)	? ($p = 2 \implies$ not (T))	?
$G \hookrightarrow$ some l^p-space, coarsely.	yes	?	?
G has finite asymptotic dimension.	yes	yes	?
Something on $asym - cone(G)$.	$\mathbb{R}$-tree	Behrstock–Druţu–Sapir	?
Out(G) finite or G splits along virtually abelian subgroups.	yes, finite virtually cyclic (Bestvina–Paulin–Rips).	Out(G) = 1 (Ivanov)	Out(G) = 1 (Bridson–Vogtmann)
(compactness theorem) $\exists$ finitely many $G \curvearrowright X_i$ hyperbolic, $1 \leq i \leq N$ such that $G \curvearrowright X_1 \times \cdots \times X_N$ is proper / QI-embed.	yes	yes	?