TOPOLOGICAL DIMENSION OF THE BOUNDARIES OF SOME HYPERBOLIC $\mathrm{Out}(F_n)$-GRAPHS

CAMILLE HORBEZ

ABSTRACT. A theorem of Bestvina–Bromberg–Fujiwara asserts that the mapping class group of a hyperbolic surface of finite type has finite asymptotic dimension; its proof relies on an earlier result of Bell–Fujiwara stating that the curve complex has finite asymptotic dimension. The analogous statements are still open for $\mathrm{Out}(F_n)$. In joint work with Mladen Bestvina and Ric Wade, we give a first hint towards this, by obtaining a bound (linear in the rank n) on the topological dimension of the Gromov boundary of the graph of free factors of F_n (as well as some other hyperbolic $\mathrm{Out}(F_n)$-graphs).

Theorem (BHW). *The Gromov boundary of the* free factor graph FF_N *has topological dimension* $\leq 2N-2$.

(intersection graph / co-surface graph $\leq 2N-3$, cyclic splitting graph $\leq 3N-5$)

Theorem (Bestvina–Bromberg–Fujiwara). Σ *oriented surface of finite type* $\rightsquigarrow$ $\mathrm{Mod}(\Sigma)$ *has finite* asymptotic dimension.

$\implies$ $\mathrm{Mod}(\Sigma)$ satisfies the integral Novikov conjecture.

Definition (Gromov). A metric space X has *asdim* $\leq n$ if $\forall R > 0, \exists$ open cover of X by subsets of uniformly bounded diameter with R-multiplicity $\leq n+1$.

That is, every R-ball intersects at most $n+1$ sets from the cover.

e.g. $\mathbb{R}^2$ has asymptotic dimension 3.

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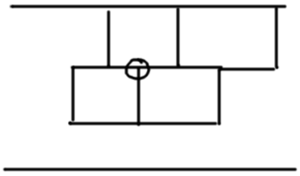


Open question. $asdim(\mathrm{Out}(F_n)) < +\infty$?

(1) (Bell–Fujiwara)

The *curve graph* $C(\Sigma)$ has finite asdim.

(2) qi embed $\mathrm{Mod}(\Sigma)$ into a finite product of hyperbolic spaces built out of $C(S), S \subseteq \Sigma$ subsurfaces.

Definition (Buyalo)**.** A metric space Z has *capacity dimension* $\leq n$ if $\exists\, C > 0$ such that $\forall\, \epsilon > 0, \exists$ open cover of Z by subsets with diameter $\leq \epsilon$ with $\frac{\epsilon}{C}$-multiplicity $\leq n+1$.

Theorem (Bestvina–Bromberg)**.** $capdim(\partial_\infty C(\Sigma)) \leq 4g + p - 4$.

(remark: $\partial_\infty(C(\Sigma)) \simeq \{$ ending laminations $\}$ by Klarreich)

$\implies$ (via Buyalo) $asdim(C(\Sigma)) \leq 4g + p - 3$.

Gabai had already bounded the $topdim(\partial_\infty C(\Sigma))$.

[Tools: train tracks + splitting sequences]

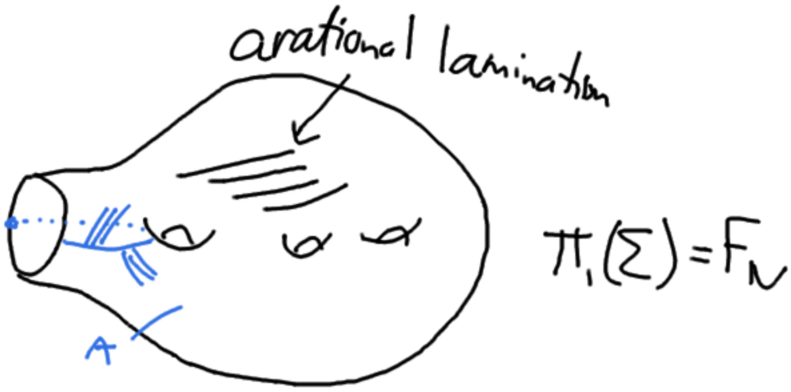
The *free factor graph* FF_N is the graph with

- vertices $\leftrightarrow$ conjugacy classes of proper *free factors* of F_N ($A \leq F_N$ such that $F_N \simeq A * B$)
- edges $[A] - [B] \leftrightarrow A \subsetneq B$ or $B \subsetneq A$.

FF_N is hyperbolic (Bestvina–Feighn).

Definition. A minimal F_N-action on an $\mathbb{R}$-tree T is *arational* if

- T is not free and simplicial
- $\forall\, A \subsetneq F_N$ free factor, $A \curvearrowright T_A$ (minimal A-invariant subtree) free and simplicial.



Theorem (Bestvina–Reynolds, Hamenstädt). $\partial_\infty FF_N \simeq \mathcal{AT}/\sim$.

Here, $\mathcal{AT}$ is the arational trees.

$T \sim T'$ if $\exists\, F_N$-equivariant alignment-preserving map $T \to T'$.

Remark. $\mathcal{AT} \subseteq \partial CV_N$

Gaboriau–Levitt: $dim(\partial CV_N) = 3N-5$.

$\mathcal{AT} \to \mathcal{AT}/\sim$ "nice"

These two facts together imply $cohodim(\partial_\infty FF_N) \leq 3N-5$.

Topological criterion.

Let X separable metric space.

(1) If $X = X_0 \cup \cdots \cup X_k$, X_i 0-dimensional $\implies$ $dim X \leq k$.

[index map $X \to \{0, \ldots, k\}$]

(2) If $X_i = \cup_{j\in\mathbb{N}} X_i^j$, each closed 0-dimensional subspaces $\implies$ $dim X_i = 0$.

Stratification of $\partial_\infty FF_N$

Definition. A *train track* is the data of

- S free and simplicial F_N-tree
- an F_N-invariant equivalence relation on $V(S)$

- for each equivalence class X of vertices, a $Stab(X)$-invariant equivalence relation on the set of directions at the vertices in X

Definition. An F_N-tree T is *carried* by τ (denoted $\tau \hookrightarrow T$) if $\exists\, f : S \to T$ F_N-equivariant such that

- $\forall\, v, v' \in V(S), f(v) = f(v') \iff v \sim v'$, and
- f identifies the germs of 2 directions d, d' at $X \iff d \sim d'$.

Index of a tt.

$$i(\tau) := \sum_{\substack{F_N-\text{orbits of equiv. classes } X \\ \text{of vertices in } \tau}} (\alpha_X + 3r_X - 3)$$

α_X = number of $Stab(X)$-orbits of directions at vertices in X

$r_X = rk(Stab(X))$

$i_{geom}(T) = \sum_{F_N\text{-orbits of branch points}} (\alpha_v + 3r_v - 3)$

$\tau \hookrightarrow T \implies i(\tau) \leq i_{geom}(T) \leq 2N - 2$ (since arational)

$i(T) = max\,\{i(\tau) \,|\, \tau \hookrightarrow T\}$

$\partial_\infty FF_N = X_0 \cup \ldots X_{2N-2}$

$X_i = \bigcup_{i(\tau)=i} P(\tau), P(\tau) = \{T \,|\, \tau \hookrightarrow T\}$.

Proposition.
- $\partial P(\tau) \subseteq \bigcup_{j>i} X_j \implies$ *$P(\tau)$ is closed in X_i*
- *$P(\tau)$ is 0-dimensional*

  $\to$ folding sequences of train tracks