COUNTING LOXODROMICS FOR HYPERBOLIC ACTIONS

SAMUEL TAYLOR

ABSTRACT. Consider a nonelementary action by isometries of a hyperbolic group G on a hyperbolic metric space X. Besides the action of G on its Cayley graph, some examples to bear in mind are actions of G on trees and quasi-trees, actions on nonelementary hyperbolic quotients of G, or examples arising from naturally associated spaces, like subgroups of the mapping class group acting on the curve graph.

We show that the set of elements of G which act as loxodromic isometries of X (i.e those with sink-source dynamics) is generic. That is, for any finite generating set of G, the proportion of X-loxodromics in the ball of radius n about the identity in G approaches 1 as n goes to infinity. We also establish several results about the behavior in X of the images of typical geodesic rays in G. For example, we prove that they make linear progress in X and converge to the boundary of X. This is joint work with I. Gekhtman and G. Tiozzo.

I.

Finitely generated $G \curvearrowright X$ by isometries.

Question. What is the dynamical behavior of a typical $g \in G$?

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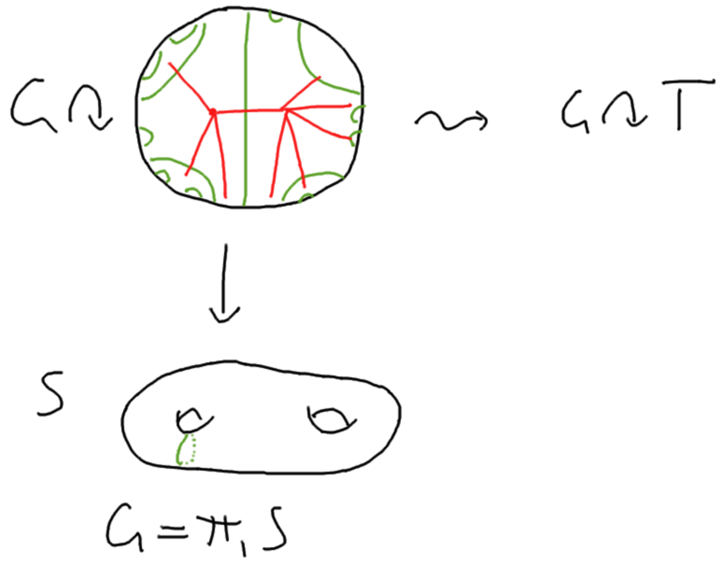


Example.

dynamics: $g \in G$

elliptic: g fixes a vertex

loxodromic: g has an axis and translates by $\tau(g)$

$\tau(g) = i([g], \alpha)$

Typical?

(1) Random walk method.

(2) Counting method: S finite, $G = \langle S \rangle$. $B(n) = \{g \in G : |g| \leq n\}$.

Definition. $P \subseteq G$. Say P is generic if

$$\frac{\#(B(n) \cap P)}{\#B(n)} \underset{n \to \infty}{\to} 1.$$

II.

- X = hyperbolic (separable) metric space.

For $g \in Isom(X)$,

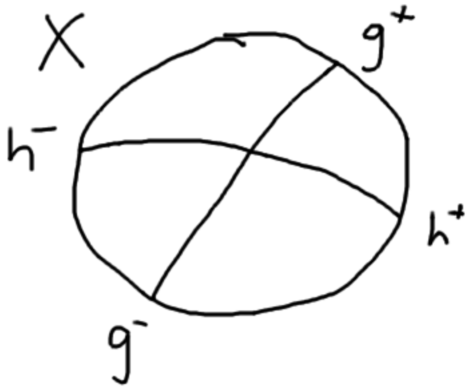
$$\tau_X(g) = \lim \frac{d_X(x, g^n x)}{n}$$

(independent of choice of $x \in X$).

g is *loxodromic* ($g \in LOX$) $\iff$ $\tau_X(g) > 0$.

- G = hyperbolic group.

- $G \curvearrowright X$ is non-elementary, i.e. $\exists\, g, h \in G$ which are independent loxodromics.



Example.

(1) $G \curvearrowright \mathrm{Cay}(G)$

(2) $G \curvearrowright$ trees / quasitrees

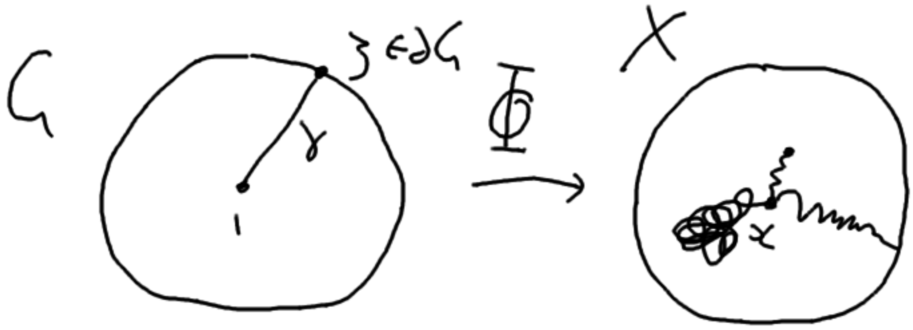
(3) $G \curvearrowright \mathrm{Cay}(H), G \twoheadrightarrow H^{\mathrm{hyp}}$

Theorem (GTT)**.** *Let G be a hyperbolic group with a nonelementary action on a separable hyperbolic space X. Then LOX is generic, i.e.,*

$$\frac{\#\{g \in B(n) \mid g \text{ is lox.}\}}{\#B(n)} \underset{n\to\infty}{\to} 1.$$

Given $G \curvearrowright X$, fix a basepoint $x \in X$. Look at the orbit map

$$\Phi : G \to X \quad g \mapsto gx$$



Given a point $\xi \in \partial G$ represented by γ, what does $\Phi(\gamma)$ look like?

Patterson–Sullivan measure on ∂G:

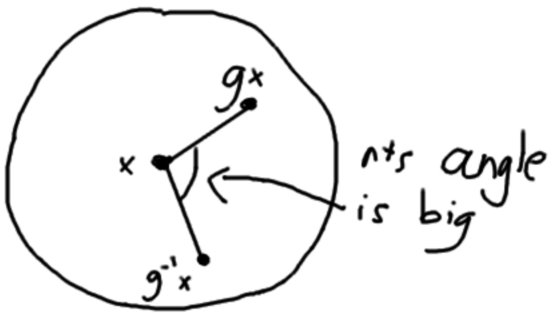
$\nu_n = \sum_{g \in B(n)} \delta_g / \#B(n)$, measure on $G \cup \partial G$.

Let $\nu = \lim \nu_n$, PS measure on ∂G.

Theorem (GTT). *Fix $x \in X$ and $G \curvearrowright X$ as above. $\exists\, L > 0$ depending only on $G \curvearrowright X$. For ν-a.e. $\xi \in \partial G$ and geodesic $(g_n)_{n \geq 0}$ with $g_n \to \xi$*

- *$g_n X$ converges to a point in ∂X*
- *such that*
$$\lim \frac{d_X(x, g_n x)}{|g_n|} = L$$
- *$\exists$ (quasi-)geodesic ray r in X such that*
$$\frac{d_X(g_n x, r)}{n} \underset{n \to \infty}{\to} 0.$$

Show:
$$\frac{\#\{g \in B(n) \mid d_X(x, gx) \geq L|g|\}}{\#B(n)} \underset{n \to \infty}{\to} 1.$$



Combining these:

Theorem. *For $G \curvearrowright X$ as above,*
$$\frac{\#\{g \in B(n) \mid \tau_X(g) \geq L|g|\}}{\#B(n)} \to 1.$$

Applications

(1) $G \curvearrowright \mathrm{Cay}(G)$

(2) $\pi_1(S) \curvearrowright T$. $\exists\, L$ such that

$$\{g \in \pi_1(S) \,|\, i([g], \alpha) \geq L|g|\}$$

is generic in $\pi_1(S)$ (α simple closed curve)

(3) Let $\phi : G \twoheadrightarrow H$ non-elementary hyperbolic groups. Then ϕ is generically bi-Lipschitz, i.e., $\exists\, L = L(G, H)$ such that

$$\{g \in G \,:\, |\phi(g)| \geq L|g|\}$$

is generic in G.

(4) $Mod(S)$ (*not* hyperbolic)

Question. Are pseudo-Anosovs typical in $Mod(S)$?

$G = Mod(S) \curvearrowright \mathcal{C}(S) = X$, and $\{\text{pA}\} = LOX$.

Answer. Pseudo-Anosovs are typical with respect to random walks (Rivin–Maher)

Open whether $\{\text{pA}\}$ is generic.

Theorem. *Let $G \leq Mod(S)$ hyperbolic (and containing 2 independent pseudo-Anosovs). Then pseudo-Anosovs are generic in G.*

G hyperbolic and cubulated $\implies G \hookrightarrow A(\Gamma)$

Theorem. generically elements map to rank 1 isometries.