RANDOM GROUPS AND LARGE-SCALE GEOMETRY

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ABSTRACT. Probabilistic methods have been used highly successfully in graph theory over the past 70 years, with two different flavors of approach. First, such methods are used to show the existence of graphs with some pathological properties that are hard to explicitly construct. Second, random or typical graphs are studied in their own right as interesting and important objects.

In Gromov's 1987 paper on hyperbolic groups, he described how many typical finitely presented groups are hyperbolic. Since then a variety of authors have studied random groups, again with the two approaches above: building exotic counter-examples (notably Gromov's construction of a finitely presented group that does not coarsely embed into Hilbert space), and the study of properties of typical finitely presented groups in a variety of models (notably Gromov's density model).

In this talk we'll survey this history and discuss some work, in part joint with Cornelia Druţu, which takes steps towards distinguishing the quasi-isometry types of random groups.

1. BUILDING STRANGE OBJECTS

Graphs:

(1) Erdős 1947: $\forall\, k$ there is a graph with $\lfloor 2^{k/2} \rfloor$ vertices so that there is no clique of size k and no independent set of size k.

Proof. Keep each edge in the complete graph $K_{\lfloor 2^{k/2} \rfloor}$ with probability $\frac{1}{2}$. Estimate: $Pr(\text{conclusion holds}) > 0$. ☐

(2) Expander graphs (Pinsker 1973).

(3) Keevash 2014: Designs.

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Remark. You don't need to think of these as random objects if you don't like, you could think of these as counting proofs.

Groups:

(1) Gromov 2000, 2003: $\exists$ f.g. group G (Gromov monster) that doesn't coarsely embed into Hilbert space.

More details in Arzhantseva–Delzant 2008.

Osajda 2014: get expanders $\hookrightarrow$ isometrically Cayley graph.

(2) Naor–Silberman 2010: Find f.g. Γ that has FL^p for all $p > 1$, and much more.

(3) I. Kapovich–Weidmann 2014: $\forall\, n \geq 2, \exists$ group of rank n, Γ, with generating sets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, so that $(a_1, \ldots, a_n, e, \ldots, e)$ and $(b_1, \ldots, b_n, e, \ldots, e)$ are not Nielsen equivalent, where both generating tuples contain $n-1$ copies of the identity e (whereas with n copies of e added, any two generating sets of size n are always Nielsen equivalent).

2. Typical objects

Graphs: Let $G_{n,m} = \{$ simple graphs with n vertices, m edges $\}$.

Let $M = M(n)$.

Say Property P holds asymptotically almost surely (a.a.s.) if

$$Pr(G \in G_{n,m(n)} \text{ has } P) \overset{n\to\infty}{\to} 1.$$

Example. For $M(n) = cn \log n$, if $c > \frac{1}{2}$ then a.a.s. G is connected, $c < \frac{1}{2}$ then a.a.s. it's not.

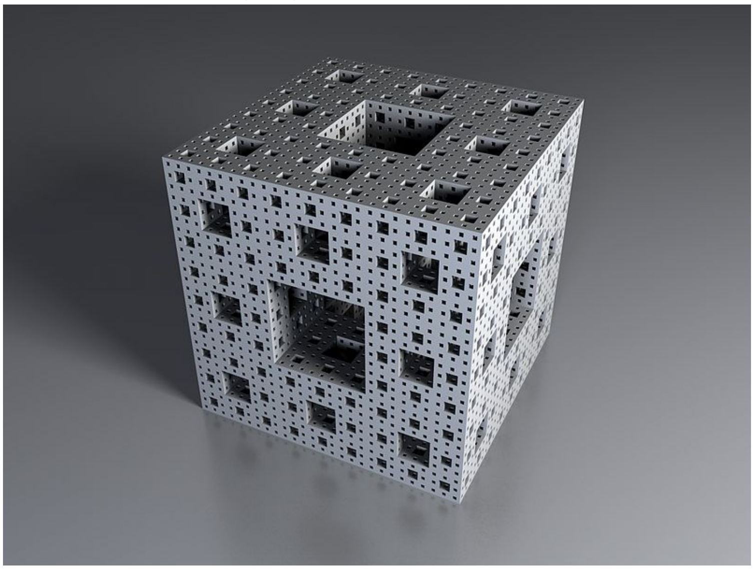
Groups:

$$\Gamma_{m,n,l} = \{ \langle s_1, \ldots, s_m \mid r_1, \ldots, r_n \rangle, |r_i| = l \text{ cyclically reduced, } \\ \text{chosen at random from } \sim (2m-1)^l \text{ possibilities} \}$$

Different choices of dependencies of m, n, c will give different models.

Few relators: Fix $m \geq 2, n \geq 1$, let $l \to \infty$.

FIGURE 1: Menger sponge, source: niabot https://commons.wikimedia.org/wiki/File:Menger-Schwamm-einfarbig.jpg



Density: Fix $m \geq 2, d \in [0,1], n = (2m-1)^{dl}, l \to \infty$.

Triangular: Fix $l = 3, d \in [0,1], n = (2m-1)^{3d}, m \to \infty$.

What's known?

Few relator (Gromov, Ol'shanskii $\sim$1987)

- a.a.s. Γ is $C'(\frac{1}{6})$ small cancellation so hyperbolic, cohomological dimension 2.

  $\partial_\infty \Gamma \cong_{homeo}$ Menger sponge (Champetier, $\sim$1995).

  Arzhantseva–Ol'shanskii: a.a.s. any $m-1$ generator subgroup is free.

Density model

- triviality: $d > \frac{1}{2}$ then a.a.s. Γ is $\{1\}$ or $\mathbb{Z}/2\mathbb{Z}$, whereas if $d < \frac{1}{2}$ then a.a.s. Γ is hyperbolic, cohomological dimension 2 (Gromov 1991, Ollivier 2003).

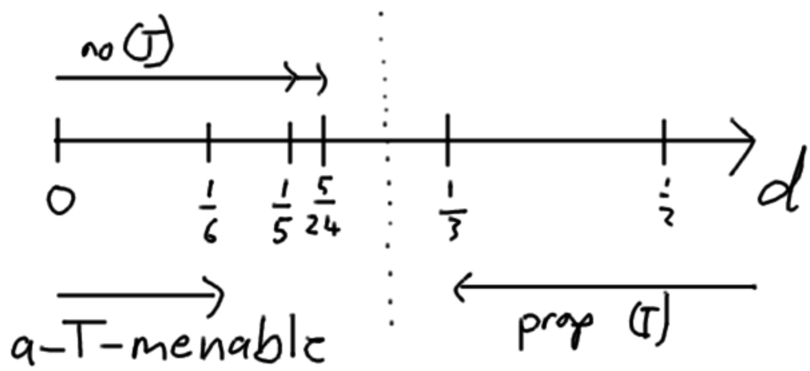
  Morally, the $\frac{1}{2}$ comes from the birthday paradox: for $d > \frac{1}{2}$ with good chance you have two relators that differ only in the first letter.

For $d < \frac{1}{12}$ then $C'(\frac{1}{6})$ i.e. no subword of length $\frac{l}{6}$ appears twice in the relations.

Proof. $Pr(\text{failure}) \leq (\#\text{ places to check})(Pr(\text{ match there }))$. This is approximately $((2m-1)^{ld}2l)^2(2m-1)^{-\frac{l}{6}} \to 0$. $\square$

Question. Sharp threshold? $(2m-1)^{\frac{1}{2}l}$.

Question (Benjamini)**.** What is the last infinite group you see as you add relations one-by-one? (You could also ask for any other monotone property, such as Property (T).)



a-T-menable due to Ollivier–Wise 2010, Property (T) due to Żuk and Kotowski–Kotowski 2012, improvement to $\frac{5}{24}$ by M. and Przytycki.

Question. Where do we switch from no (T) to (T)?

Other motivations: testing group, e.g., surface subgroups.

Question. Residual finiteness? For $d < \frac{1}{6}$, yes, but for $d > \frac{1}{3}$ unknown.

3. What do they look like?

That is, what can we say about these up to QI?

- hyperbolic, $\partial_\infty \Gamma \cong_{homeo}$ Menger ($d < \frac{1}{2}$ Dahmani–Guirardel–Przytycki 2010)

How do we know that this isn't always the same group? We can't tell up to QI, but at least up to isomorphism we see that

$$\chi(\Gamma) = 1 - m + (2m-1)^{ld}$$

grows.

Definition (Pansu)**.** Given Γ hyperbolic group. Let

$$\text{Confdim}(\partial_\infty \Gamma) = \inf\{Q \mid \exists\, X \simeq_{QI} \Gamma, \text{visual metric on } \partial_\infty X, \\ \text{measures } \mu \text{ on } \partial_\infty X \text{ with } \mu(B(z,r)) \approx r^Q\}.$$

Facts.

- $\dim_{top} \partial_\infty \Gamma \leq \text{Confdim}(\partial_\infty \Gamma) < \infty$
- QI invariant of Γ

Example. $\text{Confdim}(\partial_\infty \mathbb{H}^4_{\mathbb{R}}) = 3$.

$\text{Confdim}(\partial_\infty \mathbb{H}^2_{\mathbb{C}}) = 4$.

Theorem (M.)**.** *If $d < \frac{1}{8}$, then a.a.s.*

$$\text{Confdim}(\partial_\infty \Gamma) \approx_{1000} \log(2m-1) \frac{d}{\log d} l.$$

Corollary. *As $l \to \infty$, change QI type.*

Corollary. *a.a.s.* $\log \chi(\Gamma) / \text{Confdim}(\partial_\infty \Gamma) \approx |\log d|$.

Now in the few relators model:

Theorem (M.)**.** *Fix $c > 0, \alpha \geq 0$, and let $n = Cl^\alpha$.*

$\text{Confdim}(\partial_\infty \Gamma) = 2 + \alpha + o(1)$.

Corollary. *For all but countably many α, as $l \to \infty$ change QI type.*

Question. Can we remove "but countably many" in the above corollary?

Theorem (Druţu–M.)**.** *Triangular model: $d \in (0,1], n = (2m-1)^{3d}, m \to \infty$.*

For $d > \frac{1}{3}$ a.a.s. have FL^p $\forall\, p \in [2, \frac{1}{C} \left(\frac{\log m}{\log \log m} \right)^{\frac{1}{2}}]$.

Corollary. *a.a.s.*

$$\frac{1}{c}\left(\frac{\log m}{\log\log m}\right)^{\frac{1}{2}} \leq p(\Gamma) \leq \mathrm{Confdim}(\partial_\infty \Gamma) \leq C \log m.$$

4. Other questions

People look at random RAAG groups, random nilpotent groups, random topology.

Question. Look at all lengths at once (Ollivier "Invitations" book). Temperature model?

Question (Kleiner)**.** QI rigidity?