Kazhdan sets in groups and equidistribution properties

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(Joint work with SOPHIE GRIVAUX)

Plan:
1. Definition of Kazhdan sets. Motivation.
2. Kazhdan sets and equidistribution properties
   [Answer to an open problem due to Y. Shalom]
3. Description of Kazhdan sets in some amenable groups — in locally cpt. abelian groups, Heisenberg groups $H_n(\mathbb{R})$, $Aff_+(\mathbb{R})$, ... [Answer to a question from book of Bekka-de la Harpe-Valette]

(4. About the proof)

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1. Kazhdan sets

G second countable, locally cpt. group

$Q \overset{\text{subset}}{\subseteq} G$, $1 \in Q$ (or Q is generating ~~for~~ G).

Def: For such a subset $Q \subset G$ and $\varepsilon > 0$, we say that (Q, ε) is a Kazhdan pair if any (strongly continuous unitary) representation π of G on a Hilbert space having a (Q, ε)-invariant vector has a non-zero G-invariant vector.

Note: Q is called a Kazhdan set.

The definition means that if $\exists\, x \in \mathcal{H}_\pi$, $\|x\|=1$, such that

$$\sup_{g\in Q} \|\pi(g)x - x\| \leq \varepsilon ,$$

then $\exists\, y \neq 0$, $\pi(g)y = y$, $\forall g \in G$.

Terminology: $$\Big(\underset{\substack{\uparrow \\ \text{Kazhdan set}}}{Q}, \overset{\curvearrowleft\ \text{Kazhdan constant}}{\varepsilon}\Big)$$

Fact: G has a *compact* Kazhdan set $\Longleftrightarrow$ G has property (T).

The definition given above applies to all groups. We do not know much about Kazhdan sets, except for property (T) groups.

Examples (1) For discrete groups G with property (T) the Kazhdan sets are exactly the generating subsets.

(2) For each G, the pair $(G, \sqrt{2})$ is a Kazhdan pair.

So Kazhdan sets will have to be between these two extreme cases.

In the book by Bekka - de la Harpe - Valette about property (T), at the end, there is a list of open questions. The following two are about Kazhdan sets:

Q1 (Y. Shalom) For a subset Q of $\mathbb{Z}$, the property of Q being a Kazhdan set is **possibly related** to the following *equidistribution* condition:

(E) $\forall\, \Theta \in \mathbb{R}\setminus\mathbb{Q}$, The sequence $\{e^{2\pi i q \Theta}\}_{q\in Q}$ is equidistributed in S^1.

Recall: A sequence $(x_n)_{n\geq 1}$ is equidistributed in S^1 if for every interval I in S^1

$$\frac{\#\{1\leq n\leq N \mid x_n \in I\}}{N} \xrightarrow{N\to\infty} \underset{\uparrow}{m}(I)$$

normalized Lebesgue measure.

Question from the audience: Does This definition depend on an ordering of The set Q?

Answer. Possibly, but for This talk The squences are going to consist of positive numbers: $n_0 = 1 < n_1 < n_2 < \dots$. The reason is That one can deal with negative numbers by choosing $n_{-k} = -n_k$, or something similar.

Q2. More generally, what are the Kazhdan sets of $\mathbb{Z}$, $\mathbb{Z}^d$, $\mathbb{R}^d$, Heisenberg group, and other infinite amenable groups?

Why would anybody be interested in studying Kazhdan sets? They were used for quantitative versions of property (T). Here we study Kazhdan sets in groups without property (T), in particular for some amenable groups. This becomes a problem about asymptotic properties of these non-compact sets.

Q "remembers" some rigidity properties that the group may have.

Theorem (implicit in an article of Burger-Ozawa-S.Thom)

If Q is a Kazhdon set in G, if $d \in \mathbb{Z}, d \geq 1$, then for every $\alpha > 0$, $\exists \beta > 0$ s.t. for every two finitely dimensional representations π_1, π_2 of G on $\mathcal{H} = \mathbb{C}^d$ that are close over Q, meaning $\| \pi_1(q) - \pi_2(q) \| < \beta$, $\forall q \in Q$,

then there exists a unitary U, $\| U - \dot{I} \| < \alpha$, such that $\pi_2(g) = U^* \pi_1(g) U$, $\forall g \in G$.

There are also other applications of Kazhdon sets in ergodic theory (see the paper with Sophie, available on arXiv).

2. Shalom's question

Consider sequences of positive integers

$$n_0 = 1 < n_1 < n_2 < \dots < n_k < \dots .$$

[If negative numbers are needed, let $n_{-k} = -n_k, k \geq 1$.]

Recall also the equidistribution property:

(E) $\forall \theta \in \mathbb{R} \setminus \mathbb{Q}$, the sequence $\left(e^{2\pi i n_k \theta} \right)_{k \geq 1}$ is equidistributed.

Theorem (B - Grivaux)

(a) For $Q = \{ n_0 = 1, n_1, n_2, \dots \}$ in $\mathbb{Z}$ we have
$(E) \Longrightarrow Q$ is a Kazhdan subset for $\mathbb{Z}$.

(b) There are sets of the above form which are Kazhdan and not verifying (E).

This will follow from a characterization of Kazhdan sets for locally compact abelian groups.

Example: The sequence $n_k = 2^k + k$, $k \geq 1$, is a Kazhdan set but it is lacunary: there exists $r > 1$ such that $\frac{n_{k+1}}{n_k} \geq r > 1$, $\forall k$.

From work of Pollington and, independently, de Mathan it is known that such lacunary sequences do not have good equidistribution properties: there are "many" (in the sense of Hausdorff dimension) irrational θ s.t. $\left(e^{2\pi i n_k \theta}\right)_k$ is not dense mod 1.

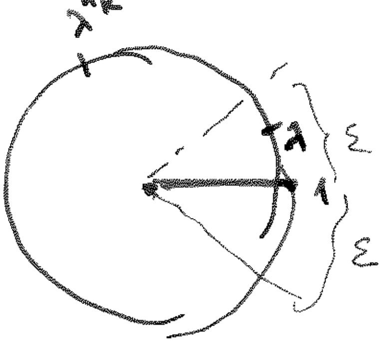
The complicated nature of Kazhdan sets is shown by the following:

Non-example: The sequence $n_0 = 1$, $n_k = 2^k$, $k \geq 0$, is not a Kazhdan set. (It is also lacunary.)

These examples can be proven either directly or using Their characterization.

There is a classical result of Weyl: $(n\Theta)_{n\geq 1}$ is equidistributed iff $\Theta \in \mathbb{R}\setminus\mathbb{Q}$. It also works for $(n^k\Theta)_{n\geq 1}$, some polynomials etc.

Example. If $\frac{n_{k+1}}{n_k} \longrightarrow \infty$ (super-lacunary), Then (n_k) is not a Kazhdan set. (So Kazhdan sets cannot grow too fast.) Consider a 1-dimensional representation, $U: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \lambda z$, $|\lambda|=1$. Having ε-invariant vector, means $|\lambda^{n_k}-1|<\varepsilon$, $\forall k$. For The operator to have a fixed point it means That $\lambda=1$.



3. Characterization of Kazhdan sets

it is a measure Theoretical characterization:

Theorem (B - Grivaux). Let G be a second countable locally compact abelian group. TFAE:

(1) $\underset{1\in Q}{Q} \subset G$ is a Kazhdan set for G

(2) $\exists\, \delta$, $0<\delta<1$, s.t. $\forall\, \sigma$ probability measure on the dual group Γ satisfying

$$\inf_{g\in Q} \left|\widehat{\sigma}(g)\right| > \delta$$

Then σ must have a discrete part.

Note: For $G=\mathbb{Z}$, $\Gamma = S^1$, the Fourier coefficient

$$\widehat{\sigma}(\underset{\substack{\uparrow\\ \text{character}}}{n}) = \int_{S^1} e^{-2\pi i n t}\, d\sigma(t).$$

We have the following classical result:

Wiener's lemma

σ has a discrete part $\iff \lim_{N\to\infty} \frac{1}{2N+1}\cdot \sum_{n=-N}^{N} \left|\widehat{\sigma}(n)\right|^2 > 0.$

Note: It is not easy to check this condition.

Example: $H_n(\mathbb{R})$ Heisenberg group

$$\begin{pmatrix} 1 & \underline{\vec{x}} & t \\ 0 & I_n & \downarrow \underline{y} \\ 0 & 0 & 1 \end{pmatrix} \xmapsto[\text{projection}]{\pi} (\underline{x}, \underline{y}) \in \\ \in \mathbb{R}^{2n}$$

Theorem (B-G) $Q \subset H_n(\mathbb{R})$ is Kazhdan if and only if $\pi(Q)$ is a Kazhdan set in $\mathbb{R}^{2n}$.

Example (of a "small" Kazhdan set in $\mathbb{R}^d$)

Let Q be a Kazhdan set in $\mathbb{Z}^d$, Then, $\forall \varepsilon > 0$, $B(0,\varepsilon) \cup Q$ is a Kazhdan set for $\mathbb{R}^d$!

Another example: $\mathrm{Aff}_+(\mathbb{R})$

$x \longmapsto ax+b$, $a>0$, $b \in \mathbb{R}$

$Q \subset \mathrm{Aff}_+(\mathbb{R})$ is a Kazhdan set iff The set

$$Q_0 = \left\{ t \in \mathbb{R} \;\middle|\; \exists\, b \in \mathbb{R} \text{ s.t. the map } x \longmapsto e^t x + b \in Q \right\}$$

is a Kazhdan subset of $\mathbb{R}$.

Example: The Kazhdan sets of $SL_2(\mathbb{R})$ are the sets Q s.t. $\overline{Q}$ is not compact.

Note: For $\mathbb{F}_2$ it is an work in progress. (Arnold, Kirillov(?)) ⟶ There is a notion of equidistribution for free groups

Note: There is an analogy between their characterization (σ has a discrete part) and a result of Bekka-Valette (1993):

> For σ-compact group G, G has (T) iff every action of G on a Hilbert space $\mathcal{H}_\pi$ with almost invariant vectors has a non-zero finite dimensional subrepresentation.