A nice trick involving amenable groups

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Random Walks on Groups

Random Walks

Let Γ be finitely generated group with a symmetric generating set S.

Consider the random walk on Γ

$$g_0 = 1 \quad g_i = g_{i-1} s_i$$

where s_i is randomly chosen element from S.

The random walk defines the co-growth sequence

$$a_n = \text{number of times the RW return to the identity after } n \text{ steps}$$

and the return probability

$$\rho_n = \frac{a_n}{|S|^n}.$$

Co-growth

The sequence a_n is very hard to compute, and this can be done explicitly only for groups with a very simple combinatorial structure.

Meta-Conjecture (Kontsevich) *For any finitely generated group, the sequence a_n is "nice"?*

This is "obviously false", since it will be a non-trivial result valid for all countable groups.

Theorem (Garrabrant-Pak) *There exists a finitely generated group $G \subset \mathrm{SL}_4(\mathbb{Z})$ such that a_n is not P-recursive.*

Spectral Radius

The limit $\lambda = \lim \sqrt[n]{\rho_n}$ is called spectral radius.

Theorem (Kesten) *The group Γ is amenable if and only if $\lambda = 1$.*

Remark *There is no known algorithm which given Γ can compute the spectral radius (even approximate it with arbitrary precision)*

Theorem (K-Pak) *There exist a finitely generated group with transcendental spectral radius, i.e., a_n is far from a "nice" sequence.*

Some Remarks

This result is a consequence of a construction of family of 4 generated groups whose spectral radii form a Cantor set.

This is possible even though there is no algorithm for computing the spectral radius of any of these groups.

There is an algorithm which produces a infinite presentation of a group with a transcendental spectral radius (modulo an unproven technical lemma). This algorithm is very very slow...

Idea behind the Construction

Idea

We start with a “small” group Γ at “modify” at infinitely many places by adding suitable groups.

Each “modification” can be done independently of the others and result in small decrease of the spectral radius.

This leads to family of groups indexed by all subsets on the natural numbers. If certain conditions are satisfied then the spectral radius will be continuous function and the image will be a Cantor set.

Marked Groups

A k-marked group is a group together with an ordered generating set of size k, i.e. a group Γ together with surjection $F_k \twoheadrightarrow \Gamma$.

The space of marked groups is has natural topology where two groups are close if large balls in the Cayley graphs are the same.

Let G_i are marked group, the product $\bigotimes G_i$ is a marked subgroup of $\prod G_i$ generated by the diagonal embedding of the generating sets. This product satisfy the usual universal property.

Main Lemma

Lemma *There exists an amenable marked group Γ and sequence of marked group G_i such that*

- $\lim G_i = \Gamma$ *in the Chabauty topology;*
- *there is exact sequence*

$$1 \to N_i \to G_i \to \Gamma \to 1$$

  with N_i is non-amenable;
- *there are almost no maps of marked group between G_i and G_j.*

It is easy to see that these conditions imply that $\lambda_{G_i} \to 1$.

Remark *These conditions implies that Γ is not finitely presented. I do not know an example where G_i are finitely presented*

The main theorem follows by the observation that we can pass to a subsequence $I \subset \{1, 2, \ldots\}$ to ensure that the groups

$$\Gamma_J = \bigotimes_{i \in J} G_i$$

for a finite set $J \subset I$ have different spectral radii.

This uses a result of Kesten that spectral radius decreases when taking extensions with non-ablian groups.

Remark *Explicitly constructing the subsequence I is only possible if one can compute (or at least approximate) the spectral radius for the groups G_J.*
In this case is possible to recursively construct an infinite set J such that the spectral radius of G_J avoids any countable set.

Some technical details

There are two natural candidates for the group Γ in the Main Lemma

- Lamplighter group

$$\Gamma = \mathbb{Z} \ltimes F_2[\mathbb{Z}] = \langle a, t | a^2 = 1, [a, a^{t^k}] = 1 \rangle$$

- (the first) Grigorchuk group, defined as a subgroup of automorphisms of binary rooted tree generated by 4 elements A,B,C and D of order 2 satisfying the condition $BCD = 1$.

Lamplighter

Take a non-amenable group N with an automorphism ϕ and an element A of order 2, such that N is generated by $\{\phi^k(A)\}$.

We can take G_i to be $G_i = \mathbb{Z} \ltimes N^{\times i}$ where $t \in \mathbb{Z}$ acts by

$$(n_1, n_2, \ldots, n_k)^t = (n_2, n_3, \ldots, n_k, \phi(n_1))$$

This becomes a marked group $a = (A, 1, \ldots, 1)$.

It is very easy to check that G_i converge to the Lamplighter as marked groups

The classical tools for computing (estimating) spectral radius only work when N is close to abelian.

G_i does not satisfy the last two properties in main lemma but this can be fixed.

Grigorchuk group

Start with a non-amenable group N generated by 4 elements A,B,C and D of order 2 satisfying the condition $BCD = 1$, satisfying additional conditions – we can take N to be virtually $\mathrm{SL}_2(\mathbb{Z}[1/2])$.

We can take G_i to be a modification of the Grigorchuk group — when we reach then i-th level instead of continuing we use the generators of the group N.

The contracting property of the Grigorchuk group imply that G_i converge as marked groups.

The groups G_i are S-arithmetic and (up to finite index) act on product of trees and hyperbolic planes. This should allow to algorithmically estimate their spectral radius.

Again, G_i does not satisfy the last two properties in main lemma but this can be fixed.