Our bundles fit in to this universal sequence,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S,z) & \longrightarrow & Mod(S \setminus \{z\}) & \xrightarrow{\phi} & Mod(S) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & \pi_1(S) & \longrightarrow & \Gamma_G^\rho & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where Γ_G^ρ is the pullback. If ρ is one to one we get $\Gamma_G^\rho = \phi^{-1}(\rho(G))$.

Proposition. (1) *Γ_G^ρ contains no $BS(p,q)$) if and only if $|\ker \rho| < \infty$ and $\rho(G)$ is purely pseudo-Anosov.*
(2) *If G has a finite $K(G,1)$ then Γ_G^ρ has a finite $K(\Gamma_G^\rho, 1)$.*

The finite kernel statement is technically frustrating, but does not change results up to quasi-isometry, so we restrict to injections. This gives the reformulation of our question

Question (Gromov; Farb-Mosher)**.** *Suppose $G \leq Mod(S)$. If G is purely pseudo Anosov, finietly generated (or finitely presented, $K(G,1)$ finite,...), is $\Gamma_G = \phi^{-1}(G)$ hyperbolic?*

Theorem (Bestvina-Bromberg-Kent-Leininger)**.** *Suppose $G \leq Mod(S)$. Then Γ_G is hyperbolic ifand only if G is purely pseudo Anosov, finietly generated, and* undistorted.

1. Some Background Notions

By analogy with Kleinian groups, Farb and Mosher define

Definition. *$G \leq Mod(S)$ is* convex cocompact *if the action on $\mathcal{T}(S)$ has a quasiconvex orbit in the Teichmüller metric.*

Theorem (Farb-Mosher, Hammenstädt)**.** *G is convex cocompact if and only if Γ_G is hyperbolic.*

Corollary. *Γ_G hyperbolic implies that G is finitely generated, purely pseudo Anosov, and undistorted.*

We remark that the theorem (BBKL) provides a converse and a $Mod(S)$ intrinsic charactarization of Γ_G hyperbolicity. An alternative characterization is given by Durham and Taylor, called stability.

2. The proof

2.1. **Tools.** The first of our tools is the *curve graph* $\mathcal{C}(S)$.

Theorem (Kent-Leininger, Hammenstädt)**.** *$G \leq Mod(S)$ is convex cocompact if and only if $G \to \mathcal{C}(S)$, the orbit map, is a quasi-isometric embedding.*

We will also use $\mathcal{M}(S)$ the marking graph of S as a model of $Mod(S)$. $\mathcal{M}(S)$ is locally finite and $Mod(S)$ acts properly discontinuously and cocompactly by simplicial isometries, the orbit map is a quasi-isometry.

The next tool is projections. Suppose $Z \subsetneq Y \subsetneq S$ are essential subsurfaces. There is a projection (due to Masur-Minsky) $\pi_Z(\mu) \subset \mathcal{C}(Z)$ for μ a marking on Y. We can also define (due to Behrstock) $\pi_{\mathcal{M}(Z)}(\mu) \subseteq \mathcal{M}(Z)$.

Proposition. *The diameters of $\pi_Z(\mu)$ and $\pi_{\mathcal{M}(Z)}(\mu)$ are bounded.*

Definition. *Given $\mu_1, \mu_2 \in \mathcal{M}(Y)$,*

$$d_Z(\mu_1, \mu_2) = diam(\pi_Z(\mu_1) \cup \pi_Z(\mu_2))$$

and $d_{\mathcal{M}(Z)}$ is similar.

Projections are used to define a distance formula.

Theorem (Masur-Minsky)**.**

$$d_{\mathcal{M}(S)}(\mu_1, \mu_2) \asymp \sum_{Y \subseteq S} [d_Y(\mu_1, \mu_2)]_M$$

2.2. **First Key Ingredient: Pigeonhole Proposition.** Suppose $G \leq Mod(S)$ finitely generated, and μ is a marking. Given $c > 0$ there exists an $R > 0$ with the following property. If $g \in G$, $Z \subsetneq S$, $|g| > R$ and $d_{\mathcal{M}(Z)}(\mu, g\mu) \geq c|g|$ then G contains a reducible element.

2.3. **Sketch of a proof of the theorem.** Suppose $G \leq Mod(S)$ is finitely generated, undistorted, purely pseudo Anosov, and torsion free. Suppose $\mu \in \mathcal{M}(S)$.

Undistorted implies

$$|g| \asymp d_{\mathcal{M}(S}(\mu, g\mu) \asymp \sum_{Y \subseteq S} [d_Y(\mu, g\mu)]_A$$

We want an $\epsilon > 0$ such that $d_S(\mu, g\mu) \geq \epsilon|g|$ for all g. So suppose not, that for every $\epsilon > 0$ there is some $g \in G$ such that $d_S(\mu, g\mu) < \epsilon|g|$. For such g there are subsurfaces $Y_1, \ldots, Y_k \subsetneq S$ such that

(1) $d_Y(\mu, g\mu) \geq A$
(2) $|g| \asymp \sum_{i=1}^k d_{Y_i}(\mu, g\mu)$

From Bestvina-Bromberg-Fujiwara and Behrstock-Kleiner-Minsky-Mosher, there exists a $0 < B < A$ and

(3) $g = g_1 \cdots g_k$, $|g| = \sum |g_i|$, $g_i \in G$
(4) $d_{Y_j}(\mu, g_1 \cdots g_i\mu) \leq B$ if $i < j$
(5) $d_{Y_j}(g\mu, g_1 \cdots g_i\mu) \leq B$ if $i \geq j$

Using this we find a subsurface with linearly large marking projection of some piece of g, producing a reducible element and a contradiction.