DILATATIONS OF PSEUDO ANOSOV MAPPING CLASSES NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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Let $S = S_{g,p}$ be the complete, oriented, connected surface of genus g with n punctures. Consider $\varphi \in Mod(S) = Homeo^+(S)/\text{isotopy}$.

Assume φ is pseudo-Anosov, that is S has a φ invariant pair of measured foliations $(\mathcal{F}^\pm, \nu^\pm)$ so that for some $\lambda(\varphi) > 1$ outside $Sing(\mathcal{F})$ $\varphi_* \nu^\pm = \lambda^\pm \nu^\pm$.

Definition. *φ is* orientable *if $\mathcal{F}$ is, and φ is essential if $Sing(\mathcal{F}) = \emptyset$.*

Question (Minimum Dilatation Problem)**.** *What is $\lambda_{min}(S) = \{\lambda(\varphi) : \varphi \in \mathcal{P}(S)\}$ where $\mathcal{P}(S) = \{\varphi \in Mod(S) : \varphi$ is pseudo Anosov$\}$.*

This question is hard. A related question involves the *Normalized Dilatation*

$$L(S, \varphi) = \lambda(\varphi)^{|\chi(S)|}$$

Question. *What is the smallest accumulation point of $L(P)$, $P = \bigcup_S \mathcal{P}(S)$?*

Conjecture is that the smallest accumulation point is μ^4 where μ is the golden ratio.

1. Why is $L(S, \varphi)$ a natural thing to study?

First $\log \lambda_{min}(S)$ is the length of the shortest geodesic on $\mathcal{M}(S)$. The results of Penner, Hironaka-Kin, Valdina, and Tsai show that as $|\chi(S)| \to \infty$ $\lambda_{min}(S) \to 1$ and also

$$\log \lambda_{min}(S_{g,p}) \asymp \frac{1}{|\chi(S)|}$$

along rays through zero in the (g,p) plane of rational slope.

Dilatations are also related to Perron numbers. Recall that λ is a *Perron number* if there is a Perron Frobenius matrix A with Perron Frobenius eigenvalue λ. The Perron Frobenius degree of λ is

$$\deg_{PF}(\lambda) = \min\{d | \text{there is a } d \times d \text{ PF matrix with eigenvalue } \lambda\}$$

A *Perron unit* is a perron number such that $\frac{1}{\lambda}$ is also algebraic.

McMullen showed that If λ is a Perron unit then $\lambda^{\deg_{PF}(\lambda)} \geq \mu^4$. In general $\deg_{PF}(\lambda(\varphi)) \leq 3|\chi(S)|$.

Notes prepared by Edgar A. Bering IV.

1.1. **Example.** Consider the map of the 4-punctured sphere induced by the braid $\sigma_1\sigma_2^{-1}$ in the standard generators. $\lambda(\varphi) = \frac{3\sqrt{5}}{2} = \mu^2$ and $|\chi(S_{0,4})| = 2$.

2. Thurston's fibered face theory

Consider the hyperbolic mapping torus $M = S \times [0,1]/\varphi$ of φ for $(S, \varphi) \in P$. We have a Z cover $q : S \times \mathbb{R} \to M$, which induces a flow $f : M \times \mathbb{R} \to M$ via $(q(x,t), s) \mapsto (x, t+s)$.

The stable foliation $\mathcal{F}$ lifts to a foliation of M. We say $(S_1, \varphi_1) \cong (S_2, \varphi_2)$ are flow equivalent if $(M_1, \mathcal{L}_1)$ is homeomorphic to $(M_2, \mathcal{L}_2)$.

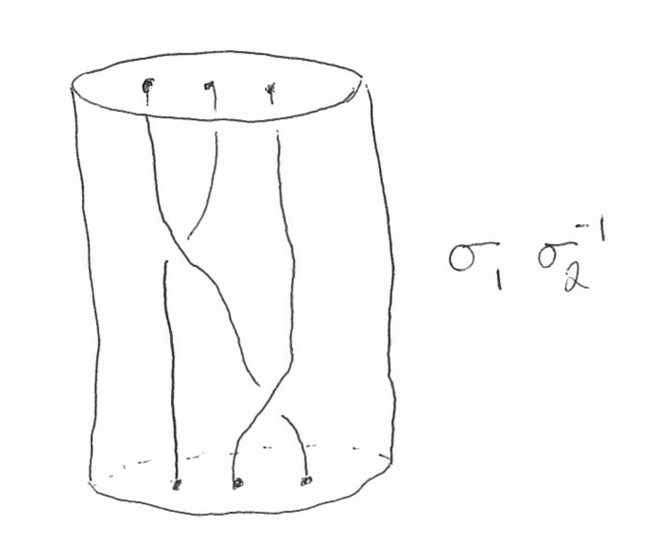
This breaks P into equivalence classes.

Given a flow class $\mathcal{C}$ and M the mapping torus, each $(S, \varphi) \in \mathcal{C}$ defines a primitive integral class in $H^1(M, \mathbb{R})$. We can see this as follows, each φ induces a foliation of M giving a map $H_1(M) \to \mathbb{Z}$. Primitivity follows from connectivity.

The natural map $\mathcal{C} \to H^1(M, \mathbb{R})$ lands on the integral priminitve elements of a cone $F \cdot \mathbb{R}^+$.

Thurston defines a norm on $H^1(M, \mathbb{R})$, which has the property that $||\alpha(S, \varphi)||_T = |\chi(S)|$. Let B_T be the unit ball in this norm. It is a convex polyhedron, and given $\mathcal{C}$ there is a top dimensional face F such that $\alpha(\mathcal{C})$ is the integral primitive elements of $F \cdot \mathbb{R}^+$. We call F the fibered face.

Fried showed that L extends to a continuous convex function on F which goes to infinity near the boundary. As a consequence, each $\mathcal{C}$ that is not isolated has $L(\mathcal{C})$ dense in $[L_{\mathcal{C}}, \infty)$ for some $L_{\mathcal{C}} > 1$. This gives us the following picture of $L(P)$



Farb-Leininger-Margalit show that for any $L_0 > 0$ there is a finite set of essential $\mathcal{C}$ such that $L_{\mathcal{C}} < L_0$.

3. Results

Theorem. *If $\mathcal{C}$ is non-isolated, essential, and contains an oriented element then $L(S, \varphi) \geq \mu^4$ for all $\varphi \in \mathcal{C}$.*

Theorem. *If β a braid on n strands defines a pseudo Anosov essential element of $P(S_{o,n+1})$ then $L(S, \beta) \geq \mu^4$.*

Lemma (Key Lemma)**.** *If $\mathcal{C}$ is non-isolated, essential, and contains an oriented element then Δ_M the Alexander polynomial and $\Theta_{\mathcal{C}}$ the Teichmuller polynomial are equal up to signs of the coefficients. In fact we can choose coordinates $u, t_1, \dots, t_k$ such that $\Delta_M(u, t_1, \dots, t_k) = \Theta_{\mathcal{C}}(\pm u, t_1, \dots, t_k)$.*

3.1. **Some Notation.** Let H be the free abelian group generated by $h_0, \dots h_k$, and $\mathbb{Z}H \cong \mathbb{Z}[u^{\pm}, t_1^{\pm}, \dots, t_k^{\pm}]$. This isomorphism is realized by the map

$$\sum a_h x^h \mapsto \sum a_h u^{m_0} t_1^{m_1} \cdots t_k^{m_k} \qquad h = m_0 h_0 + \cdots + m_k h_k$$

For $\alpha \in Hom(H, \mathbb{Z})$ define $\theta^{(\alpha)}(x) = \sum a_h x^{\alpha(h)}$. We get an associated norm on $Hom(H, \mathbb{R})$, $||\alpha||_\theta = \max\{|\alpha(h_1) - \alpha(h_2)| \, | h_1, h_2 \in Supp(\theta)\}$.

From McMullen we know

$$||\alpha||_{\Delta_M} \leq ||\alpha||_\Theta \qquad \text{for all } \alpha \in F \cdot \mathbb{R}^+$$

In our case, the lemma tells us

$$|\chi(S)| = ||\alpha||_T = ||\alpha||_{\Delta_m} = ||\alpha||_\Theta = \deg_{PF}(\lambda(\varphi)$$

where the equality of Thurston and Alexander polynomial norms is due to McMullen. This equation then implies the first theorem by Perron Frobenius theory.

3.2. **Some remarks on the proof of the Key Lemma.** Let $\mathcal{L}$ on M, lift to the maximal abelian covering $\tilde{\mathcal{L}}$ on $\tilde{M}$. Consider the Teichmuller module $T(\tilde{\mathcal{L}})$, Θ_M is the primitive generator for the first fitting ideal for $T(\tilde{\mathcal{L}})$. The Alexander polynomial plays an identical role in $H_1(\tilde{(}M)$. We consider both as $\mathbb{Z}H$-modules where H is the deck group of the maximal covering.

For a fixed $(S, \varphi) \in \mathcal{C}$, the foliation $\mathcal{F}$ gives a presentation of the module $T(\mathcal{L})$. And a train track representative for $\mathcal{F}$ gives a presentation for cohomology, and all things fit together in the following diagram

$$\begin{array}{ccccc}
T(\tilde{\mathcal{F}}) & \longrightarrow & T(\tilde{\mathcal{F}}) & \longrightarrow & T(\tilde{\mathcal{L}}) \\
\uparrow & & \uparrow & & \uparrow \\
R^e & \longrightarrow & R^e & \longrightarrow & H^1(\tilde{M}, \mathcal{O}) \\
\uparrow & & \uparrow & & \uparrow \\
R^v & \longrightarrow & R^v & \longrightarrow & H^0(\tilde{O})
\end{array}$$

It follows that

$$\Theta = \frac{\det(uI - \tilde{\varphi}_e)}{\det(uI - \tilde{\varphi}_v)} \qquad \Delta = \frac{\det(uI \pm \tilde{\varphi}_e)}{\det(uI \pm \tilde{\varphi}_v)}$$