A NEW BOUNDARY FOR THE MAPPING CLASS GROUP NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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This work is joint with Hagen and Sisto. Plan

- Some HHSes
- Gromov ∂
- Main Theorem
- Subgroup ∂
- $Mod(S)$ and $PML(S)$
- Masur-Minsky Theory
- $\partial Mod(S)$
- (4) again
- Toward geometric finiteness

1. Some HHSes (Behrstock-Hagen-Sisto)

- Hyperbolic Spaces
- $\mathbb{R}^n$ products of HHSes
- All cubical groups (BHS, Hagen-Suisse) (eg RAAGs and RACGs)
- $\pi_1(M^3)$ with M^3 closed and no Nil or Sol components (BHS)
- $Mod(S)$ and $T(S)$ with either the Teichmuller or Weil-Peterssen metrics. (Masur-Minsky, Brock, Rafi, Durham, Augab, Behrstock,...)

2. Gromov Boundary

Given X a delta hyperbolic space define $\partial_{gr}X$ to be asymptotic classes of geodesics based at a point.

Theorem (Gromov)**.** *If X is proper (eg a group) then $\partial_{gr}X$ is compact and metrizable.*

Things you can do with ∂_{gr}:

- Classification of elements by dynamics on ∂_{gr}
- Analyse structure of subgroups
- $\partial_{gr}G$ is a model for the Poisson boundary four random walks
- Cannon-Thurston maps
- Geodesic flow spaces
- Patterson-Sullivan theory

Notes prepared by Edgar A. Bering IV.

3. Main Theorem

Theorem (Durham-Hagen-Sisto). *For any HHS X there exists a bordification ∂X such that if X is proper then ∂X is compact and metrizable. If X has a group action the action extends continuously.*

Examples. Hyperbolic groups recover the Gromov boundary. RA(A/C)Gs retopologizes the simplicial boundary.

Things we can do with ∂G

- Nielsen-Thurston like classification of elements
- "rank one" elements act with North-South dynamics on ∂G
- ∂G is a compact model for the Poisson boundary
- A Tits alternative
- A Rank-Rigidity theorem a la Caprace-Sageev
- Handel-Mosher Omnibus Subgroup Theorem

Theorem (Handel-Mosher,Durham-Hagen-Sisto). *If $G < Mod(S)$ let $A(G) = \bigcup_{g \in G} supp(g)$. Then there exists $g_0 \in G$ such that $supp(g_0) = A(G)$.*

4. Boundaries of Subgroups

Let $H < G$ be groups with "nice boundaries" ∂H ∂G. Natural questions

(1) Does there exist an H equivariant continuous map $\partial H \to \partial G$?
(2) Does there exist an H equivariant continuous extension of $i : H \to G$ to $\partial i : \partial H \to \partial G$? (These are called Cannon-Thurston maps).
(3) Is the above map an embedding?

5. $Mod(S)$ and $PML(S)$

$Mod(S)$ acts on $T(S)$ the Teichmüller space of S properly and by isometries but this action is not cocompact.

- $Mod(S)$ is not quasi-isometric to $T(S)$
- Dehn twists are distorted

Thurston showed that $PML(S)$ is a boundary, $Teich(S) \cup PML(S)$ is compact, and the $Mod(S)$ action extends continuously.

But, Lenzhen showed that there are Teichmüller geodesics which limit to full simplices of $PML(S)$.

6. Masur-Minsky Theory

Consider $\mathcal{M}(S)$ the marking graph of S. This is quasi-isometric to $Mod(S)$. Let

$$P_Y = \{\mu \in \mathcal{M}(S) | \partial Y \subset base(\mu)\}$$

for $Y \subset S$,

$$P_Y \cong \mathcal{M}(Y) \times \mathcal{M}(S \setminus Y) \times \prod_{\alpha \in \partial Y} \mathbb{Z}$$

This set P_Y quasi-isometrically embeds in $\mathcal{M}(S)$, is an infinite product region.

Therefore any "nice" boundary should see a simplicial join of boundaries of components of P_Y.

7. $\partial Mod(S)$

Define $p \in \partial Mod(S)$ by its support *supp*(p) a collection of pairwise disjoint subsurfaces and a formal linear combination

$$p = \sum_{Y \in supp(p)} \alpha_Y \cdot \lambda_Y$$

such that $\sum \alpha_Y = 1$ and $\lambda_Y \in \partial\mathcal{C}(Y) \cong \mathcal{EL}(Y)$, where this isomorphism is due to Klarreich.

Theorem (Durham-Hagen-Sisto)**.** *There exists a topology on $\partial Mod(S)$ which makes $Mod(S) \cup \partial Mod(S)$ compact and metrizable.*

- $\partial P_Y \hookrightarrow \partial Mod(S)$ embed
- $\partial \mathcal{C}(S) \hookrightarrow \partial Mod(S)$ is full measure in any lifting measure.
- Suppose $\mu_n \to p$ such that *supp*(p) $= \{Y_1, Y_2\}$. Fix $\mu \in \mathcal{M}(S)$ and suppose $p = \alpha_1\lambda_1 + \alpha_2\lambda_2$. Then

$$\lim_{n\to\infty} \frac{d_{Y_1}(\mu, \mu_n)}{d_{Y_2}(\mu, \mu_n)} = \frac{\alpha_1}{\alpha_2}$$

8. Subgroup ∂ revisited

Theorem (Durham-Hagen-Sisto)**.** *Let $G < Mod(S)$ be any of the following*

1. *$Mod(Y)$ for $Y \subseteq S$*
2. *Convex cocompact*
3. *A finite coarea Veech subgroup V*
4. *Leininger-Reid combinations of (3)*

Then $i : G \to Mod(S)$ extends G equivariantly to an embedding $\partial i : \partial G \to \partial Mod(S)$.

Theorem (Leininger-Reid)**.** *There exists $H \to Mod(S)$ such that $H = \pi_1(S')$, S' closed and all but one non identity conjugacy class of elements is pseudo Anosov.*

For (3) and (4) above the embedding does not extend to $PML(S)$.

9. Towards Geometric Finiteness

Definition (Bowditch)**.** *Suppose M is a metrizable compactum. We say G acting on M is a* convergence group action *if the action of G on $M^{(3)}$ the space of distinc triples is properly discontinuous. The action is* geometrically finite *if every point in M is a conical end point or bounded parabolic point for the action of G.*

The point of this definition is that it is very general.

Definition (Proposal)**.** *We say* $G < Mod(S)$ *is* geometrically finite *if* $\Lambda(G)$ *is compact in* $\partial Mod(S)$ *and the* G *action on* $\Lambda(G)$ *is a geometrically finite convergence group action.*

Question. *Are Veech and Leininger-Reid subgroups geometrically finite?*

Question. *If* $G < Mod(S)$ *is geometrically finite is* E_G *the surface group extension hierarchically hyperbolic?*

Not all subgroups admit Cannon-Thurston maps

Theorem (Mousely)**.** *Let* $G < Mod(S)$ *be a Koberda RAAG such that the supports of two generators don't fill. Then* G *does not admit a Cannon-Thurston map into* $\partial Mod(S)$*.*