MSRI - CAT(0)

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cubical geometry via hyperbolicity

- Existence of Factor systems (w/ T. Susse)
- Boundaries of cube complexes (w/ Durham + Sisto)

X = CAT(0) c.c.

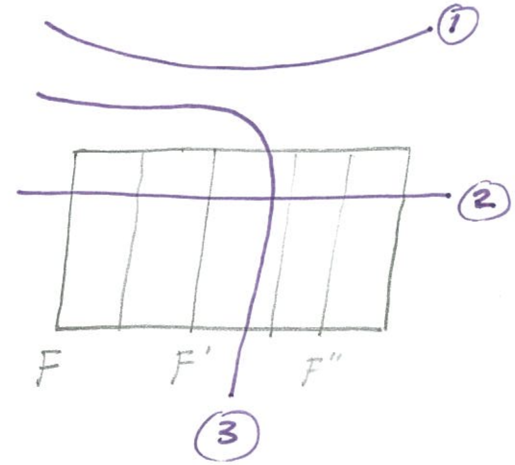
Def^{n}: $Y \subset X$ is convex provided

(1) Y is full "(no missing cubes)"

(2) 1-skeleton $Y^{(1)} \subset X^{(1)}$ is convex.

F, F' are parallel when they intersect a common hyperplane.

Given a convex $F \subset X$, the convex hull of $\{F' : F \parallel F'\} = [F]_{\parallel}$ is a product $F \times \underbrace{F^{\perp}}_{\text{orthogonal complement.}}$



hyperplanes may be

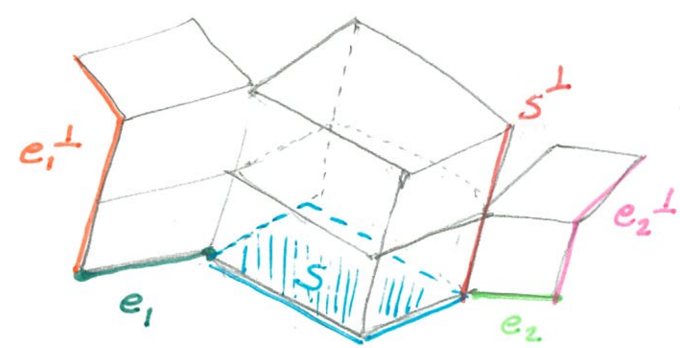
(1) disjoint from $[F]_{\parallel}$

(2) cross $[F]_{\parallel}$

(3) separate $[F]_{\parallel}$

Hull$([F]_{\parallel})$ = (2) × (3)

example:



Def$^{\text{n}}$ the <u>GATE MAP</u> $\mathfrak{g}_F : X \to F$

is given by closest point projection.

<u>Fact</u>: can cut up subcomplex into parts.
to understand orthogonal complement via projection.

in prev. example:

$$(e_1 \cup S \cup e_2)^{\perp} = \mathfrak{g}_{S^{\perp}}\left(\mathfrak{g}_{e_2^{\perp}}(e_1^{\perp})\right)$$

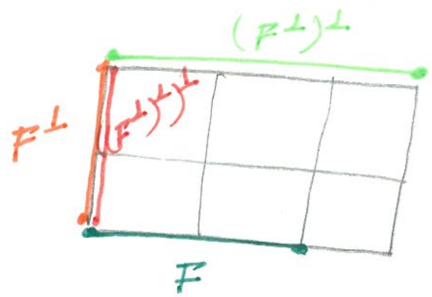
<u>Proposition</u>: $F, F' \subset X$ both convex.

① $F \subset F' \Rightarrow (F')^{\perp} \subseteq F^{\perp}$ (up to parallelism)

may need to move so they coincide.

② $((F^{\perp})^{\perp})^{\perp} = F^{\perp}$

illustration



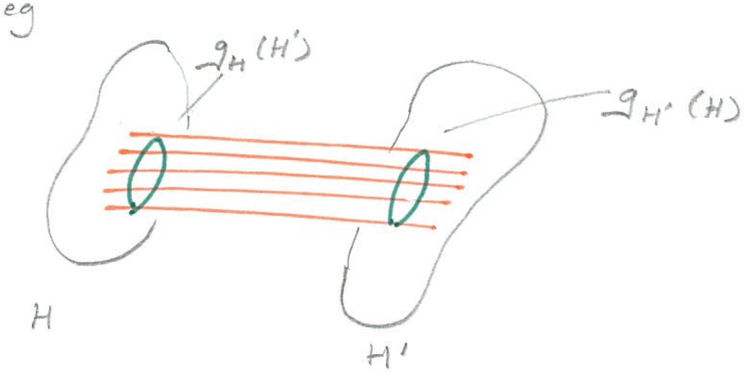
<u>N.B</u> Do not nec. have $F \,/\!/\, (F^{\perp})^{\perp}$

<u>Q</u>: What is the factor system in this setting?

<u>Candidate system</u>

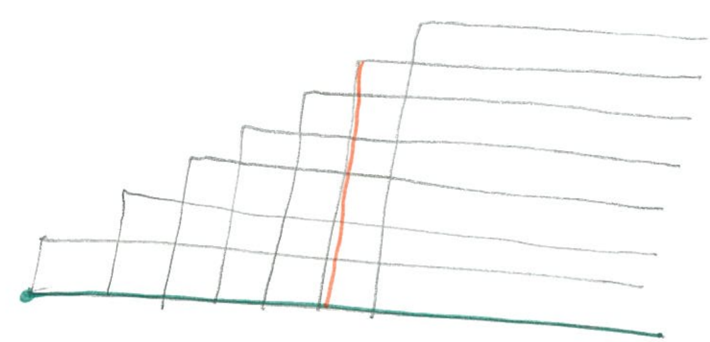
$\mathcal{F}$ = Closure under gates {combinatorial hyperplanes}

eg



* reminiscent of "bridges"

Recall from Alessandro's talk
poison subcomplex: "staircase"



Alternate characterization of $\mathcal{F}$

Proposition: $F \in \mathcal{F}$ is convex

iff $\exists$ a compact convex set

$C \subset X$ s.t. $F = C^{\perp}$

sketch//

(1) $F \in \mathcal{F} \Rightarrow F = g_{H_n} \circ \dots \circ g_{H_2} \circ g_{H_1}(H_0)$

$= g_{H_n}(F')$

(2) $F' = (C')^{\perp}$ with C' compact, convex

(3) $C = \mathrm{Hull}(C' \cup e)$

closest edge to C (adjacent)

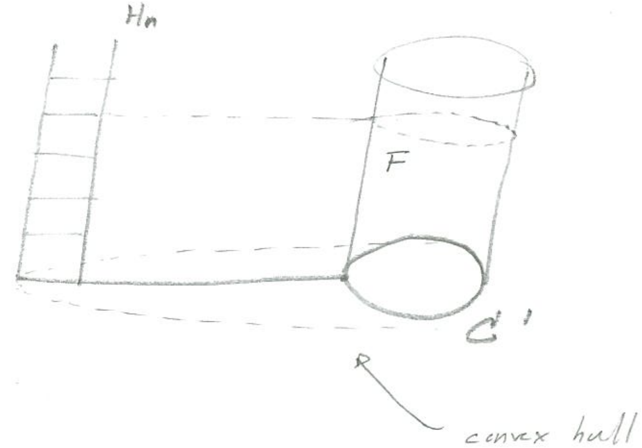
dual to H_n.

Hagen + Susse: $G \curvearrowright X$ proper + cocompact

$\Rightarrow \mathcal{F}$ is unif. loc. fin.

so is a factor system.

Corollary: X contains no convex staircases.



Boundaries.

$C(X) = \begin{cases} V = \text{hyperplanes of } X. \\ E = (\text{pairs NOT separated by another hyperplane}) \end{cases}$

- make flag

if $F \subseteq X$ is convex.

then $C(F) \subseteq C(X)$

induced subgraph.

if $F' \subseteq X$ and $F' \parallel F$

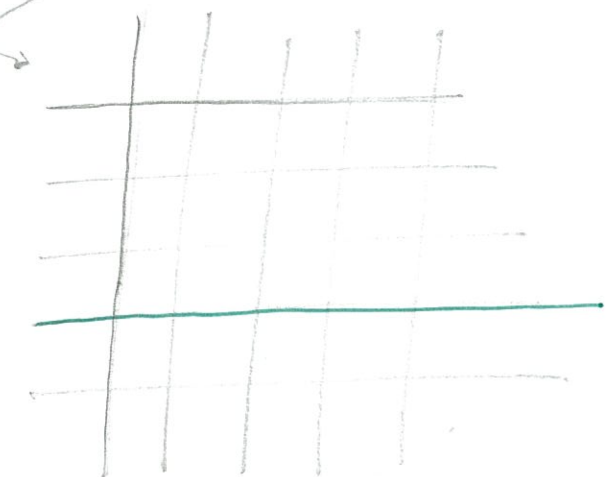
$C(F') = C(F)$

$\hat{C}(X)$ = coned $C(X)$ where we cone all $F \in \mathcal{F}$

s.t. $F \subsetneq X$

example

$X =$ standard cubing of $\mathbb{R}^2$

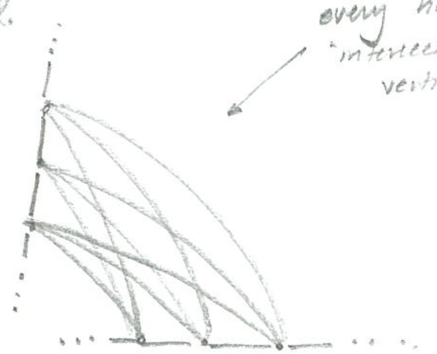


F

- Have 2-parallelism classes
  - horizontal
  - vertical.

every horizontal intersects every vertical.

$\mathcal{C}(X) =$



contact graph

for subcomplex

$\mathcal{C}(F) =$

$\mathfrak{F}$ = Factor system / parallelism.

the boundary is the simplicial complex

$$\Delta X = \begin{cases} V = \bigsqcup_{F \in \mathfrak{F}} \partial \widehat{\mathcal{C}}(F) \\ \text{simplex } [v_0, \dots, v_n] \text{ whenever } v_i \in \partial \widehat{\mathcal{C}}(F_i) \end{cases}$$

isomorphic to simplicial boundary as complexes

where $\underbrace{\prod_{i=0}^{n} F_i}_{\text{see product region inside } X.} \hookrightarrow X.$

Durham + Hagen + Sisto :

$\exists$ a (horrible) topology on ΔX s.t. it is compact + Hausdorff, $\partial_{\mathfrak{F}} X$ with topology

and (1) every simplex embeds

(2) $\partial \widehat{\mathcal{C}}(F)$ embeds ($\forall F \in \mathfrak{F}$)

(3) $\mathrm{Aut}(X) \curvearrowright \partial_{\mathfrak{F}} X$ by homeomorphisms.

Rk : X : hyp $\Rightarrow \partial_{\mathfrak{F}} X =$ Gromov. indep of HHS structure

$X = MCG \Rightarrow \partial_{\mathfrak{F}} X$ is different analogous to Caprace + Sageev.

Thm: $G \curvearrowright X$ geometric with factor system $\mathcal{F}$ has either

(1) $\exists\, g \in G$ acting loxodromically on $\hat{\mathcal{C}}(X)$

(2) $\exists$ fin. G-orbit in $\mathcal{F} - \{X\}$

G: preserves a product region.

$G \curvearrowright X$ essentially

(2) $\Rightarrow$ $X = F \times F^{\perp}$