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Cocompactly cubulated Artin groups

w/ J. Huang + P. Przytycki

Q: Which Artin groups have fin. index subgroups that act properly and cocompactly on CAT(0) cube complex

the Artin group associated to a graph, $\Gamma$, with edge labels in $\mathbb{Z}_{\geq 2}$ has presentation:

$$A(\Gamma) = \langle v \in V(\Gamma) \mid \underbrace{v \cdot w \cdots}_{n} = \underbrace{w \cdot v \cdots}_{n} \rangle$$

$$\Leftrightarrow \quad \overset{v}{\bullet}\underset{n}{\text{———}}\overset{w}{\bullet} \in E(\Gamma)$$

Examples:

RAAG: edge labels $= 2$.

Braid group: edge labels $2, 3$.

the Coxeter group corresponding to $\Gamma$

new relations

is $W_\Gamma = A(\Gamma) / \langle\langle v^2 : v \in V(\Gamma) \rangle\rangle$

the DIMENSION of $A(\Gamma)$ is the maximal cardinality subset of generators for which the induced Coxeter group is finite.

(equiv. $\dim(A(\Gamma)) = \dim(\text{Davis complex}))$
$\widetilde{W_\Gamma}$ "

In particular, $A(\Gamma)$ has $\dim \leq 2$ when $\forall$ triples $a, b, c \in$ (generators)

$$\frac{1}{m_{ab}} + \frac{1}{m_{bc}} + \frac{1}{m_{ac}} \leq 1$$

$m_{xy} =$ label of $(x, y)$.

Main Thm:

$A(\Gamma)$ has dim $\leq 2$

$\Rightarrow$ TFAE.

(1) $A(\Gamma)$ acts properly and cocompactly on a CAT(0) cube complex

(2) $\exists H \leq_{\bullet} A(\Gamma)$ s.t. $H$ acts properly and cocompactly on a CAT(0) cube complex

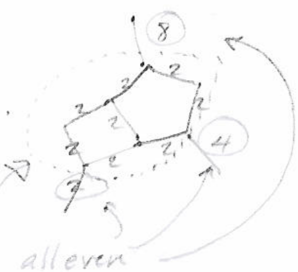
(3) Each connected component of $\Gamma$ is one of:

- vertex
- edge
- leaf edges are even and all others are 2.

eg

$\bullet$

$\bullet \xrightarrow{m} \bullet$




this is a RAAG

- Can weaken to 3-generated
  - need to allow component


Haettel: • extend to all dimensions

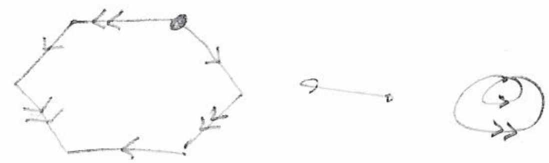
- lose statement about fin. index subgroups
- more options for components.

proof (sketch)

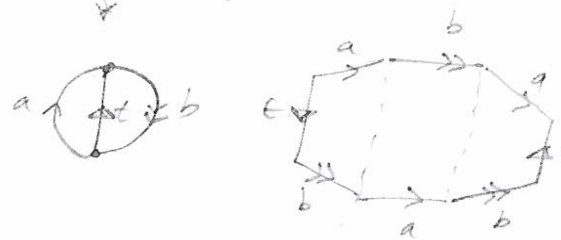
Ex 1: $\Gamma = \bullet\overset{3}{\text{—}}\bullet$

$A(\Gamma) = \langle a, b \mid aba = bab \rangle$

presentation complex.

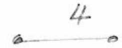


transform




all links are flag

!!
2

**Ex2:** $T =$ 


$A(T) = \langle a, b \mid abab = baba \rangle$

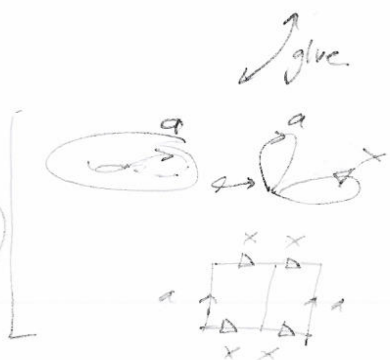
$= \langle a, \underset{x}{\underset{\|}{(ab)}} \mid ax^2 = x^2 a \rangle$

again can check that all links are flag.

**Ex3:**

$T =$ 


example 2




Rk: condition (3) follows from curvature considerations.

(1) $\Rightarrow$ (3)

main lemma:

$X$ loc. fin. CAT(0) c.c. of asymptotic rank $n$ — largest $n$ s.t. $\mathbb{R}^n \xrightarrow{q.i.} X$.

$H < Aut(X)$ s.t.

- $H = H_1 \times \ldots \times H_n$ $H_i$ all hyperbolic
- the orbit map $h \mapsto hx$ is q.i embedding.

$\Rightarrow$ $H$ is convex cocompact

Further here product decomp. of complex $\longleftrightarrow$ decomp of group.

if $H_i$'s are not virtually $\mathbb{Z}$

then $\exists$ convex subcomplex

$$Y = Y_0 \times \prod_{i=1}^{m} Y_i \quad \text{(cubical decomp.)}$$

allowed to be virt. $\mathbb{Z}$

s.t. $Y$ is $H$-cocompact.

- the induced action $\prod_{i=m+1}^{n} H_i$ ($H_i$ not virtually $\mathbb{Z}$) on $Y_0$ is proper and cocompact.

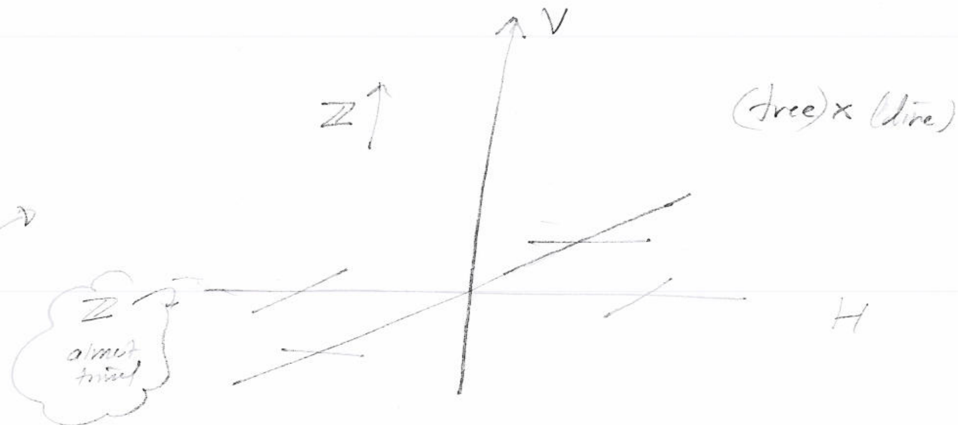
$Y_0 \simeq \mathbb{R}^{n-m}$

Further, the induced action

$H_i \curvearrowright Y_j \qquad i \neq j$

is almost trivial $\qquad j \in [n]$.

illustration




<u>Example</u> :

Suppose $A(\overset{m}{\underset{a\quad b}{\circ\!-\!\circ}})$ acts

on a CAT(0) cube of asy. rank = 2

$A = \langle a, b \mid aba = bab \rangle$

is a f.i. subgp of

$A \underset{\circ}{\leq} A' = \underbrace{\langle ab^{-1}, a^{-1}b \rangle}_{\text{free}} \times \underbrace{\langle (ab)^3 \rangle}_{\mathbb{Z}}$

so

$\exists$ convex subcomplex $Y = V \times H$ s.t.

- the induced action $\mathbb{Z} \curvearrowright V$ is proper, cocompact.
- the induced action $\mathbb{Z} \curvearrowright H$ is almost trivial

to prove (1) $\Rightarrow$ (3) rule out

poison subgraphs.

- $\Lambda = \overset{3}{\underset{a}{\circ}}\!\!-\!\!\underset{b}{\circ}\!\!\overset{3}{-}\!\!\underset{c}{\circ} \subset \Gamma$.

Assume for contradiction

$A(\Gamma) \curvearrowright X$ is proper + cocompact.

$A_{ab}$ contains subgrp

$A'_{ab} = F \times \mathbb{Z}$.

$b^6 \in \mathbb{Z}\,[F, F]$

let $Y_{ab} = V_{ab} \times H_{ab}$. (from Lemma decomp)

$Y_{bc} = V_{bc} \times H_{bc}$ ( " )

$A'_{ab} \cap A'_{bc} \cong \langle b \rangle$

the coarse intersection $Y_{ab}$ and $Y_{bc}$ = $Y_b$

cimax

$\langle b \rangle$ - cocompact

Decompose

$$\begin{tikzcd} Y_b = \arrow[d] & V_b \times \arrow[d] & H_b \arrow[d] \\ Y_{ab} & V_{ab} & H_{ab} \end{tikzcd}$$

since $b^6 \in Z[F,F]$

$V_b$ is quasi-line.

$\Longrightarrow$ $H_b$ is bounded, so

$\langle b^6 \rangle \underset{\text{commensurable}}{\sim} \langle Z \rangle \quad \underset{\text{contradiction}}{\cancel{\Rightarrow}}$