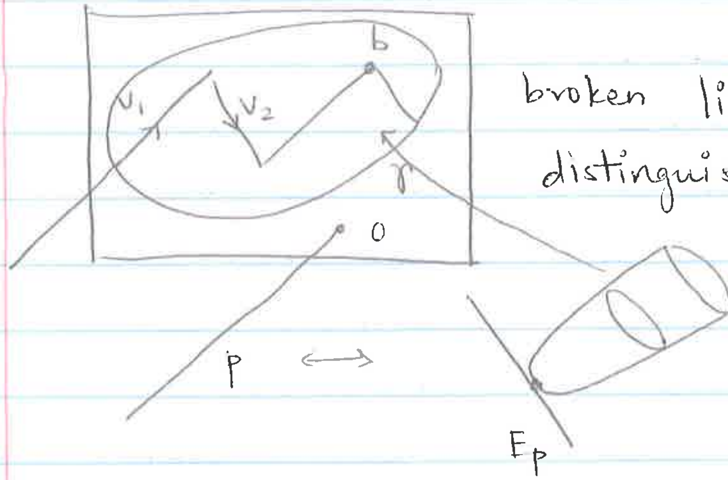
For $\mathcal{U}$ log CY, $\mathcal{U}^{trop}(\mathbb{R}) \cong \mathbb{R}^N$

piecewise lin.

$U \subset Z = U \cup E_p$



broken lines

distinguished paths

Fix $U \subset Y$ compact &.

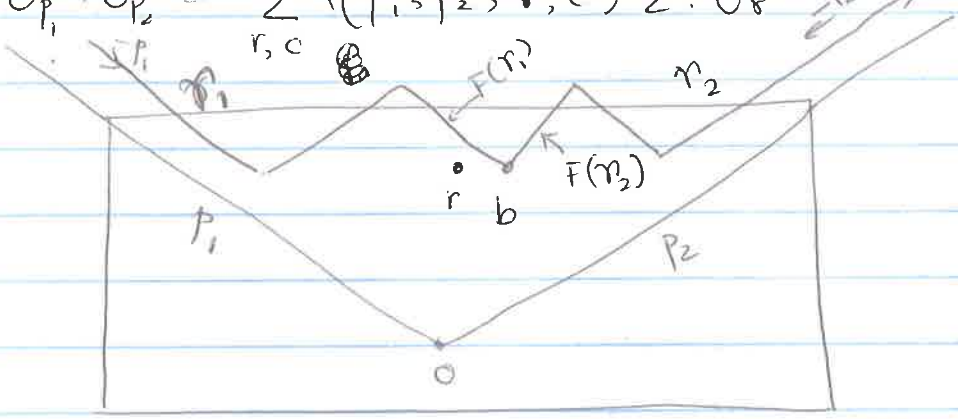
$G \subset NE(Y) \subset H_2(Y, \mathbb{Z})$ G: curve class

$m_q(\gamma) \in \mathbb{Z}$ count the discs

$$A = \bigoplus_{q \in U^{trop}(\mathbb{R})} R \cdot \theta_q$$

$$R = \bigoplus_{G \in NE(Y)} \mathbb{Q} \cdot z^G$$

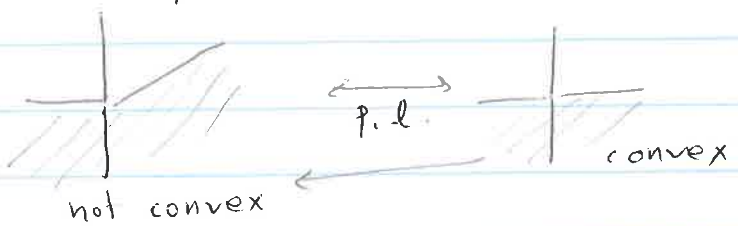
$$\theta_{p_1} \cdot \theta_{p_2} = \sum_{r, c} \alpha(p_1, p_2, r, c)\, z^c\, \theta_r$$



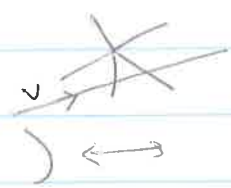
$$F(\gamma_1) + F(\gamma_2) = -r$$

$$\alpha(p_1, p_2, r, C) = \sum_{\substack{r_1, r_2 \\ \alpha_1, \alpha_2 \in NE(Y) \\ \alpha_1 + \alpha_2 = C \\ I(r_i) = -p_i \\ F(r_1) + F(r_2) = -r}} m_{\alpha_1}(r_1)\, m_{\alpha_2}(r_2)$$

"convex" polytops in $U^t = U^{trop}$



(from Paul's talk)



Easy: $f : \mathbb{R}^N \to \mathbb{R}$ p.l.

f is min-convex (min of

$Df(V)$ is non-increasing along any straight line.

$f : \mathcal{U}^{trop} \to \mathbb{R}$

Def say its "min convex" if $Df(V_i)$ is non increasing along any broken line

"In some seed" $\mathcal{U} \supset T_L$

$\mathcal{U}^{trop} \cong L_{\mathbb{R}} = \mathbb{R}^N$

Straight in some seed $\Rightarrow$ broken

$\Rightarrow f$ is min convex $\Rightarrow$ ordinary min convex in every seed

$\not\Leftarrow$

"convex cone" $\{u \in U^+ \mid f(u) \geq 0\}$

f is min convex.

"convex polytop" Fix constant $f \geq C$.

Bounded ? iff $f(u) < 0 \ \forall \ u \neq 0$.

Fock - Goncharov Conj:

A^{trop} parametrized a basis for $\mathcal{O}(X) = \mathcal{O}(\mathcal{U}^{\vee})$

$\mathcal{U}^{trop} \cong \mathbb{Z}^n$

(Imposible in general)

Need an assumption of sort:

$X \hookrightarrow A$ is affine

In Our construction

$$\underset{\substack{\uparrow \\ \text{makes sense}}}{\Theta_{p_1} \Theta_{p_2} = \sum \alpha(p_1, p_2, r)\, \Theta_r}$$

in general it is infinite sum

When do we get finite sum?

Lemma: Suppose $\alpha(p_1, p_2, r) \neq 0$, ~~f mix~~

f is min convex then

$$f(r) \geq f(p_1) + f(p_2) = \text{constant}$$

So if we have $f < 0$, then

$$\Theta_{p_1} \Theta_{p_2} = \quad r \in \underbrace{f \geq \text{constant}}_{\text{bounded}}$$

let $F : \mathcal{U} \longrightarrow \mathbb{A}^1$ rat'l fun.

$F^t : \mathcal{U}^t \longrightarrow \mathbb{R}$

$F^t(u) = u(F)$

$F^t(E) = \operatorname{ord}_E(F)$

Conj: F reg $\Rightarrow F^t$ is min convex

Use geom def^n of broken lines to prove this.

When is $F^t < 0 \iff F$ has pole on every
∂ Divisor E

easy to show if U is affine, such F exists

FG is true if U is affine
if $\exists$ boundary convex polytop.

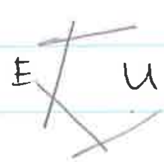
U^{trop} generelizes lattices of co-char of torus

f min convex $\Rightarrow$ $\tilde{A} \subset A[T]$
R-basis
$\vartheta_\gamma \cdot T^s, \quad f(\gamma) \geq -s$

~~tm~~
mult shows that $\tilde{A}$ graded sub-alg.
$\mathrm{Proj}(\tilde{A}) \supset \mathrm{Spec}(A)$
$(X, E) \supset \check{V}$ = mirror family

So F a reg fun on U gives P.M.M on
$V \subset \mathcal{X} = \mathcal{X}(F)$ (partial compl of mirror)
P.M.M of U
$U \subset Y$

$E \in U^t \longrightarrow \vartheta_E^{\in A} \cdot V \longrightarrow A'$



Potential fun $W_{U\subset Y} = \sum_{E \subset \partial Y} \vartheta_E$

which reg fun on mirror.

Often $U \subset Y$, Y is more interesting

e.g. $\underset{\substack{\uparrow \\ \text{open Bruhat cells}}}{U} \subset G = Y$ (care more about G)

$B_U = (U^\vee)^t(\mathbb{Z})$

$B_Y = B_U \cap \mathcal{O}(Y) \subset \mathcal{O}(U)$

Conj: $B_Y = W^t_{U \subset \cancel{X}} \geq 0$

$\subset B_U = (U^\vee)^{trop}$

$U^t \times V^t \longrightarrow \mathbb{Z}$

$(u,v) \quad u \rightsquigarrow \mathcal{O}_u : V \longrightarrow \mathbb{A}^1$

$\quad\;\; {}^{\shortparallel} \qquad\qquad v(\mathcal{O}_u) = \mathrm{ord}_E(\mathcal{O}_u)$

$u(\mathcal{O}_v)$

Conj: They are same

True in dim 2

& some cluster cases.

Assume the conj. is true.

$W^t = \{v \mid v(W) \geq 0\}$

$= \{v \mid v(\mathcal{O}_E) \geq 0\}$

$= \{v \mid \mathrm{ord}_E(\mathcal{O}_v) \geq 0\} = B_Y$

let $U \subset Y$ T: torus acting on Y preserving U

$\Rightarrow V \xrightarrow{w} T^*$

$V^t \longrightarrow X(T)$

$v \rightsquigarrow \mathcal{O}_v \leftarrow$ T-eigenfunction.

$\therefore$ w takes v to T-wt of $\mathcal{O}_v$

$V \xrightarrow{W_{ucy}} \mathbb{A}^1$

Convex polytop $(w^t)^{-1}(\ell) \cap (W^t_{ucy} \geq 0)$
$\uparrow$ $\subset B_Y$

Basis of $H^0(Y, \mathcal{O})^\ell \circlearrowleft T$

$U' \hookrightarrow Y'$ smooth proj. var.

$D \in |-K_{Y'}|$ assume D supports ample

$$\begin{tikzcd} U \arrow[r, hook, "PMM"] \arrow[d] & Y \arrow[d] \\ U' \arrow[r, hook] & Y' \end{tikzcd}$$

log CY

Y = univ. Torsor

$T_{Pic(Y')^*} = T$ $\quad \mathcal{O}(Y) = Cox(Y)$

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad \bigoplus_{L \in Pic(Y')} H^0(Y'; L)$

Conj: Have B_Y canonical basis

Let $V = U^\vee$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "W_{ucy}"'] & T_{Pic(Y)} \\ \mathbb{A}^1 & \end{tikzcd}$$

$\leftarrow$ mirror family to U

$$\begin{tikzcd} V^t \arrow[r] \arrow[d, "w^t"'] & Pic(Y) \\ \mathbb{R} & \end{tikzcd}$$

$\ni L$ line bdl

$Q(L) \subset B_U = V^{trop}(\mathbb{Z})$
"convex polytop" inside "lattice"

$\overline{\text{can. basis of } H^0(Y', L)}$ parametrize ex: G , $\mathcal{B}$ = Full flags.

$(\mathcal{B} \times \mathcal{B} \times \mathcal{B})/G = \mathrm{conf}_3(\mathcal{B})$

$$\begin{array}{l} Y = (\mathcal{A} \times \mathcal{A} \times \mathcal{A})/G \\ \downarrow \quad \downarrow H^{\times 3}\text{-Bundle} \\ Y' = (\mathcal{B} \times \mathcal{B} \times \mathcal{B})/G \end{array}$$

H - max torus.

$H^0(\mathcal{A}, \mathcal{O}) = \bigoplus_\lambda V_\lambda$

$L \in \mathrm{Pic}(Y') \longleftrightarrow$ 3 wts α, β, γ for H.

"convex polytope"

$\# Q_{(\alpha,\beta,\gamma)}(\mathbb{Z})$ = parametrizes basis for $(V_\alpha \otimes V_\beta \otimes V_c)^G$

$\shortparallel$

$C_{\alpha,\beta,\gamma}$

littlewood Rich. numbers.

By FG Y has clust struct.

each $T \subset U$ identifies $Q_{\alpha,\beta,\gamma}$ with polytop in $L_{\mathbb{R}}$.

Thm: with right choice of seed,

$W_{UC\gamma} \longleftrightarrow$ ~~support of~~ super pot of FG

$\Rightarrow Q_{\alpha,\beta,\gamma}$ Polytop of Knutson-Tao.