3/30/2016

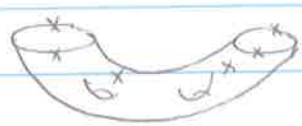
Donaldson-Thomas transformations for moduli spaces of G-local systems on surfaces

1. Moduli spaces of (framed) G-local systems

- decorated surface

$(S, \quad m)$

closed oriented top. surface w/ or w/o boundary — a finite subset of S



$m = m_b \sqcup m_p$

pts on the boundary — points inside "punctures"

- G connected, split semisimple alg gp e.g. $G = PGL_m$

flag variety $\mathcal{B}_G = G/B$
$= \{\text{Borel subgps}\}$

$\mathcal{L} \in \mathrm{Hom}(\pi_1(S-m), G)$

G-local system

$\mathcal{L}_B = \mathcal{L}_G \times \mathcal{B}_G$ associated bundle

framed G-local system

$(\mathcal{L}, \{B_i\}_{i\in m})$

B_i: a flat section of $\mathcal{L}_B$ near $i \in m$

in parti, if $i \in m_p$. B_i is invariant under g_i

$\mathcal{X}_{G,\hat{S}} := \{\text{framed G-local systems}\} / \mathrm{Inn}\, G$

Examples:

1) $\hat{S} = D_n$ disc with n-marked pts on Boundary

$$\mathrm{Conf}(B) = G \backslash B^n := \mathcal{X}_{G, D_n}$$

2) $G = PGL_2$

$$\mathcal{X}_{G,\hat{S}} \sim M^b(PGL_2(\mathbb{C}), \hat{S})$$

{Hitchin system + data}

3) $\mathcal{X}_{G,\hat{S}}$ is a positive space

$\mathcal{X}_{G,\hat{S}}(\mathbb{R}_{>0}) :=$ Higher Teichmüller space.

Assumptions:

1) $G = PGL_m$

2) $\hat{S}$ satisfies

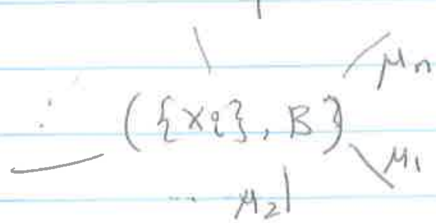
i) each boundary comp of S has at least one marked pt.

ii) $\# m \geq 1$

iii) $2\,\mathrm{genus}(\hat{S}) + 1 \geq 3$

Thm: (Fock - Goncharov) $\mathcal{X}_{G,\hat{S}}$ admits a canonical cluster Poisson structure (i.e. it is a cluster $\mathcal{X}$-variety)

Remark: Each cluster $\mathcal{X}$-variety has a set of local coordinates parametrized by seeds



$B^T = -B \leftrightsquigarrow$ quivers without loops or two-cycles

$\{\text{vertices}\} = I$ & $[b_{ij}]_+ = \max(b_{ij}, 0)$

$= \#$ arrows i & j.

There is a natural Poisson struct. encoded by B

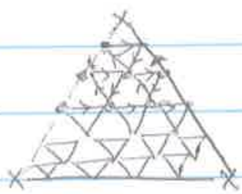
$\{x_i, x_j\} = 2 b_{ij} x_i x_j$

$(\{x_i\}, B) \xrightarrow{\mu_k} (\{x_i'\}, B')$

$\{x_i, x_j\} = 2 b_{ij} x_i x_j \iff \{x_i', x_j'\} = 2 b_{ij}' x_i' x_j'$.

Quivers for Conf_3

$G = PGL_m$



Thm (Goncharov–S) The tropicalization of (Conf_3 + superpotential) $\Longrightarrow$ Knutson–Tao Hives

The same works for all G.

Remark: The proof didnot use cluster theory. Use geometric Satake equ correspondance

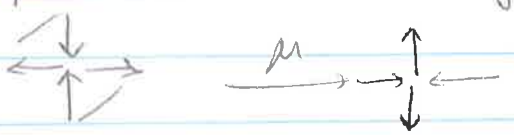
$\{\text{MV-cycles}\} \longleftrightarrow \{\mathrm{Conf}_3^{\,wt \geq 0}\}$

Quives for $\mathcal{X}_{G,\hat{S}}$:

- choose an idear triangulation $\mathcal{T}$
- replace each triangle by 

$\Rightarrow$ get a quiver $Q_{\mathcal{T}}$

If J & J' are two ideal triangulations then Q_J & $Q_{J'}$ are equivalent.
They are connected by 2-by-2 move



2. Natural actions on $\mathcal{X}_{G,\hat{S}}$

- Γ_S : mapping class gp of $S - m$

$$\begin{array}{c}\Gamma_S \\ \circlearrowright \\ \mathcal{X}_{G,\hat{S}}\end{array}$$

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G) \circlearrowright \mathcal{X}_{G,\hat{S}}$$

e.g. $G = PGL_m$, $\mathrm{Out}(G) = \mathbb{Z}/2$ if $m > 2$

$* : g \longmapsto (g^{-1})^t$

- birational Weyl gp action

g_p : monodromy

Fact : If g_p is generic then $\{B \mid g_p \cdot B = B\}$ is W-torsor
(i.e. W acts on the set)

$(\mathcal{L}, \{B_i\}_{i \in m})$

for $w \in W$, $p \in m_p$

$$(\mathcal{L}, \{B_i\}_{i \in m}) \longmapsto (\mathcal{L}, \{B_i'\})$$

$B_i' = B_i$ if $i \neq p$

$B_p' = w(B_p)$

e.g. $G = PGL_m$ $W = S_m$ symm gp.

A choice of g_p-invariant $B \iff$ ordering of eigenlines.

Summary:

$K := (\Gamma_S \times Out(G)) \ltimes W^{m_p} \circlearrowright \mathcal{X}_{G,\hat{S}}$

Goal: study K.

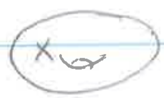
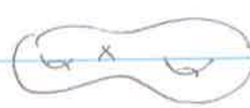
Thm: (Fock-Goncharov): $\Gamma_{\hat{S}}$ acts on $\mathcal{X}_{G,\hat{S}}$ by cluster transformations

(i.e. 1. $\Gamma_{\hat{S}}$ preserves Poisson structure of $\mathcal{X}_{G,\hat{S}}$

2. Can be presented by a seq. of class of mutations (or sometimes permutations))

Remark: All birational maps satisfying 1. & 2. above form a discrete group called the cluster modular group $\mathcal{G}$. & $\Gamma_{\hat{S}} \subset \mathcal{G}$

Assumption: If S has no boundaries, then $\# m \geq 2$

e.g. exclude  or 

Thm: (Goncharov-S) K acts by cluster transformation

i.e. $K \subset \mathcal{G}$

Remarks: 1) $G = PGL_2$, $Out(G) = 1$

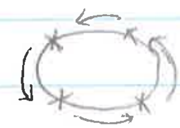
The cluster nature of $\mathbb{Z}/2$ is proved by Fock-Goncharov.

If $G = PGL_2$, $(\mathbb{Z}/2)^{m_P}$-action corresponds to tagged ideal triangulation.

2) For Conf_3, the $*$-action from $\mathrm{Out}(G)$ is the Schützenberger involution of Gelfand patterns (related to Cactus group)

3) There is a nice element of K

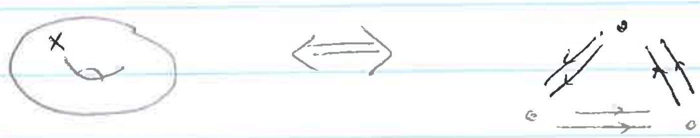
$$C_{G,\hat{S}} := (\underbrace{r_S}_{\text{cyclic shift by 1 for each comp.}}, \underbrace{*}_{\text{non-id}}, \underbrace{\{w_0\}}_{\text{longest elt}}) \in K$$



$C_{G,\hat{S}} \in \mathrm{Center}(\mathcal{G})$

Thm $C_{G,\hat{S}}$ is the DT-transformation for $\mathcal{X}_{G,\hat{S}}$

Ex $\hat{S}$:



(not covered in the thm

& $C = w_0$ is DT

(due to personal conversation bet Goncharov - Kontsevich)

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Name: Maitreyee Kulkarni   Email/Phone: mkulka2@lsu.edu

Speaker's Name: Linhui Shen

Talk Title: Donaldson-Thomas transformations for moduli spaces of G-local systems on surfaces

Date: 3/31/16   Time: 10:00 am/ pm (circle one)

List 6-12 key words for the talk: ______

Please summarize the lecture in 5 or fewer sentences: ______

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3/31/2016

Donaldson-Thomas transformations for moduli spaces of G-local systems on surfaces

1. Quantization of cluster X-variety (Fock-Goncharov)

We have $(\Lambda, (*,*))$

Lattice, skew symm form

$\longrightarrow$ Quantum torus algebra T_Λ, a $\mathbb{Q}(q)$-free module with basis $\{X_v, v \in \Lambda\}$

product relation: $X_v \cdot X_w = q^{(v,w)} X_{v+w}$

$$\lim_{q \to 1} \frac{X_v X_w - X_w X_v}{q-1} = 2(v,w) X_{v+w}$$

(recovers the Poisson structure)

Geometric quiver: without loops or two cycles

isom. classes of

B skew-symm matrices $\Longleftrightarrow$ $(\Lambda, (*,*), \{e_i\}) : Q$

$b_{ij} = (e_i, e_j)$

$\downarrow$

$T_Q \cong T_\Lambda$ with basis $\{X_i := X_{e_i}\}$

- Quiver mutations:

$$Q' = \mu_k(Q) = (\Lambda, (*,*), \{e_i'\})$$

$$e_i' = \begin{cases} -e_k & i = k \\ e_i + [(e_i, e_k)]_+ e_k & i \neq k \end{cases}$$

$$[a]_+ = \max\{a, 0\}$$

It recovers the mutation of B

Let $i_k : T_Q \cong T_\Lambda \cong T_{Q'}$

Rem: μ_k is not the only natural way to recover the mutation of B

$$\mu_k^-(e_i) := \begin{cases} -e_k & i=k \\ e_i + [-(e_i,e_k)]_+ e_k & i \neq k \end{cases}$$

$\mu_k^{\pm} \rightsquigarrow \Phi_k^{\pm}$ (from Tom's talk)

- Quantum dilogarithm series $\Psi(x)$

It is the unique formal power series over $\mathbb{Q}(q)$

1) starting with 1
2) $\Psi(q^2 x) = (1+qx)\Psi(x)$

$$\Psi(x) = \frac{1}{(1+qx)(1+q^3x)(1+q^5x)\cdots}$$

$$\Psi(x) = \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{|GL_n(\mathbb{F}_{q^2})|}$$

$$= \exp\left(\sum_{n\geq 1} \frac{(-1)^{n+1}}{n(q^n - q^{-n})} x^n\right)$$

- Quantum cluster mutation

$$\Phi_k := \mathrm{Ad}_{\Psi(X_k)} \circ i_k : \mathrm{Frac}(T_{Q'}) \longrightarrow \mathrm{Frac}(T_Q)$$

$$= \mathrm{Ad}_{\Psi(X_k^{-1})^{-1}} \circ i_k^-$$

Fact 1) The lim $q \to 1$ recovers the mutation of x variable
2) $\Phi_k^2 = \mathrm{Id}$

2. Donaldson-Thomas transformations (Kontsevich-Soibelman)
(Keller, Nagao, ...)

(Q, W) Quiver with potential

$\downarrow$

Ginzburg DG-algebra (path alg with ∂w)

$\rightsquigarrow$ Rep

$C(Q,W)$ 3d CY category

$\rightsquigarrow$

$(K_0(C), (*,*), [S_i])$

lattice $\qquad (*,*) = \sum_{i=0}^{3} (-1)^{i+1} \operatorname{rk} \operatorname{Ext}^i \qquad [S_i]$ simple objects

Fact: The triple recovers the quiver Q

T_Λ with algebraic generators $\{X_{[S_i]}\}$

$\widehat{R_{Q,q}}$ = { formal power series starting with 1 }

$= \left\{ 1 + \sum_{\alpha \in \Lambda^+} f_\alpha X_\alpha , \quad f_\alpha \in \mathbb{Q}(q) \right\}$

$\Lambda^+ = \bigoplus \mathbb{Z}_{\geq 0} [S_i]$

$\downarrow$

this is a group & Center$(\hat{R}) = 1$

Stability condition:

$S : K_0(C(Q,W)) \longrightarrow \mathbb{C}$ such that

$S[S_i] \in \mathbb{H}$ 

DT-invariants: $\forall \gamma \in \Lambda^+ \subset K_0(C)$

$\Omega_s(\gamma)$: "virtual count of semistable objects with class γ"

$\rightsquigarrow$

quantum version $\Omega_s(\gamma) \in \mathbb{Q}(q)$

Thm (Kontsevich - Soibelman) [Wall-crossing formula]

All DT can be packaged in a power series $\in \widehat{R}_{Q,q}$

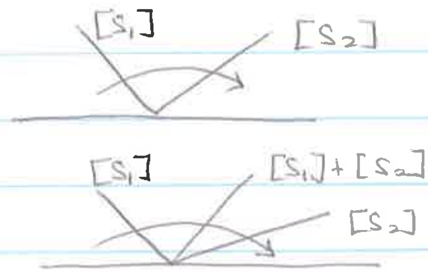
$$E_Q = \overrightarrow{\prod_{\ell:\text{rays}\ \subset \mathbb{H}}} \left(\prod_{\substack{r \in \Lambda^+ \\ s(r) \in \ell}} \Psi(X_r)^{\circ \Omega_s(r)} \right)$$

Here $\Psi(x) = \exp\left(\sum_{i \geq 1} \frac{(-1)^i}{q^i - q^{-i}} x^i \right)$

$$\Psi(x)^{\circ \Omega(q)} = \exp\left(\sum_{i \geq 1} \frac{(-1)^i \Omega(q^i)}{q^i - q^{-i}} x^i \right)$$

- E_Q is independent of the choice of stability
- Given E_Q, can recover all $\Omega_s(\gamma)$

Ex. $\underset{1}{\circ} \longrightarrow \underset{2}{\circ}$



$$E_Q = \Psi(X_{S_1}) \Psi(X_{S_2})$$
$$= \Psi(X_{S_2}) \Psi(X_{S_1+S_2}) \Psi(X_{S_1})$$

- Mutation of quivers with potential

$$(Q, W) \rightsquigarrow (Q', W') = \mu_k(Q, W)$$

$$\underbrace{(K_0, (*,*), [S_i])}_{\Lambda_Q^+} \xrightarrow{\mu_k} \underbrace{(K_0, (*,*), [S_i'])}_{\Lambda_{Q'}^+}$$

Consider the intersection

$$\widehat{R}_{Q,q} \cap \widehat{R}_{Q',q} := \left\{ 1 + \sum_{\alpha \in \Lambda_Q^+ \cap \Lambda_{Q'}^+} f_\alpha X_\alpha \right\}$$

Thm: $\Psi(X_{S_k})^{-1} E_Q = E_{Q'} \Psi(X_{S_k'})^{-1} \in \widehat{R}_Q \cap \widehat{R}_{Q'}$ (*)

DT transformation : $D_Q := \mathrm{Ad}_{E_Q} \circ \Sigma$

$\Sigma(X_v) = X_v^{-1}$

Facts : 1. D_Q is not necessarily rational

2. If D_Q is rational, get $D_Q : \mathrm{Frac}(T_Q) \xrightarrow{\cong} \mathrm{Frac}(T_Q)$

3. From (*), get a commutative diag

$$\begin{tikzcd} \mathrm{Frac}(T_Q) \arrow[r, "DT_Q"] \arrow[d, "\Phi_k"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Frac}(T_Q) \arrow[d, "\Phi_k"] \\ \mathrm{Frac}(T_{\mu_k(Q)}) \arrow[r, "DT_{\mu_k(Q)}"] & \mathrm{Frac}(T_{\mu_k(Q)}) \end{tikzcd}$$

As $q \to 1$,

$$\begin{tikzcd} \mathbb{Q}((\mathbb{C}^*)^n) \arrow[r, "DT_Q"] \arrow[d, "\mu_k"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Q}((\mathbb{C}^\times)^n) \arrow[d, "\mu_k"] \\ \mathbb{Q}((\mathbb{C}^\times)^n) \arrow[r] & \mathbb{Q}((\mathbb{C}^\times)^n) \end{tikzcd}$$

$\Rightarrow$ unique DT for cluster X-variety

Examples 1. For $\mathcal{X}_{G,\hat{S}}$, $DT = C_{G,\hat{S}} = (r_s, *, \{w_0\})$

2. Grassmannian $Gr(k,n)$

$\rightsquigarrow$ $\mathrm{Conf}_n(\mathbb{P}^m)$ cluster X-variety

"generalized pentagram map"

$p : (l_1, l_2, \ldots, l_m) \longmapsto (h_1, \ldots, h_m)$

$h_k := (l_1, \ldots, l_{k-1})$ hyperplane in $\mathbb{P}^m$.

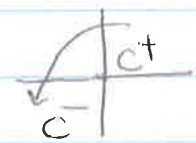
Thm (D. Weng) $P = DT$ for $\mathrm{Conf}_n(\mathbb{P}^m)$

& $DT^{2n} = \mathrm{Id}$

Tropicalization of DT and Duality Conjecture

$DT^t : \mathcal{X}_Q(\mathbb{Z}^t) \longrightarrow \mathcal{X}_Q(\mathbb{Z}^t)$

$DT(\mathbb{Z}^n_{\geq 0}) = \mathbb{Z}^n_{\leq 0}$



If DT is a cluster transformation

then $C^+, C^- \subset \Delta^+$ - cluster complex.

By (GHKK) : $\mathcal{A}_{SL_m, \hat{S}}(\mathbb{Z}^t)$ parametrizes a basis of $\mathcal{O}(\mathcal{X}_{PGL_m, \hat{S}})$

$\mathcal{A}_{SL_m, \hat{S}}$ - moduli space.

$\square$