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Generic bases are dual semicanonical bases for unipotent cells

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1. Overview: (dual) (semi-) canonical basis and cluster algebras

C : symmetrizable Cartan matrix

$\mathfrak{g}$: corres. Kac-Moody Lie algebra

$\mathcal{U}_q(\mathfrak{g})$: quantization of $U(\mathfrak{g})$

early 1990's Kashiwara & Lusztig constructed canonical lower-global basis B_q of $\mathcal{U}_q(\mathfrak{n}) \subset \mathcal{U}_q(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

B_q has many surprising properties:

$\rightsquigarrow$ basis for each integrable highest weight rep of $\mathcal{U}_q(\mathfrak{g})$

Lusztig's methods yield (in symm) case) also surprising positivity properties:

Structure constants are in $\mathbb{N}[q, q^{-1}] \subset \mathbb{Q}(q)$

Later (2000) Lusztig interrelated semicanonical basis $\varphi \subset \mathcal{U}(\mathfrak{n})$ for symm case;

this basis is constructed in terms of preprojective algebras using constructible functions on their representation spaces.

$\varphi \cap B_1 = ?$

Leclerc: $\varphi \neq B_1$ except for very small cases

In a series of papers, we used dual semicanonical
$\varphi^* \subset \mathbb{C}[N] = U(n)^*_{gr}$

$\rightsquigarrow$ Showed that many varieties coming from Lie theory (unipotent cells) have a coordinate ring which is spanned by a part of φ^* & has a cluster algebra structure with $\{\text{cluster monomials}\} \subset \varphi^*$

Recent remarkable progress: $\varphi^* \cap B_1^* \supset \{\text{cluster monomials}\}$

In fact: $B_q^* \supset \{q\text{-cluster monomials}\}$

2. Preprojective algebras and dual semican. basis

Restrict for the Dynkin case & $w = w_0$ (ADE)

$\rightsquigarrow$ $\mathbb{C}[N]$ itself is a cluster algebra.

Running example: A_4 : $1 - 2 - 3 - 4$

$$N = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \subset SL_5(\mathbb{C})$$

- Preprojective algebra:

$$\begin{tikzcd} 1 \arrow[r, shift right, "\overline{a_1}"'] & 2 \arrow[l, shift right, "a_1"'] \arrow[r, shift right, "\overline{a_2}"'] & 3 \arrow[l, shift right, "a_2"'] \arrow[r, shift right, "\overline{a_3}"'] & 4 \arrow[l, shift right, "a_3"'] \end{tikzcd}$$

relations:
- $-a_1\overline{a_1}$
- $\overline{a_1}a_1 - a_2\overline{a_2}$
- $\overline{a_2}a_2 - a_3\overline{a_3}$
- $\overline{a_3}a_3$

This yields an algebra Λ with Λ-mod being 2CY ~~≅~~

Representation varieties $\Lambda_d = \{M(b) \in \mathbb{C}^{d(tb)\times d(hb)}\}_{b \in B_q}$ fulfill relations (*).

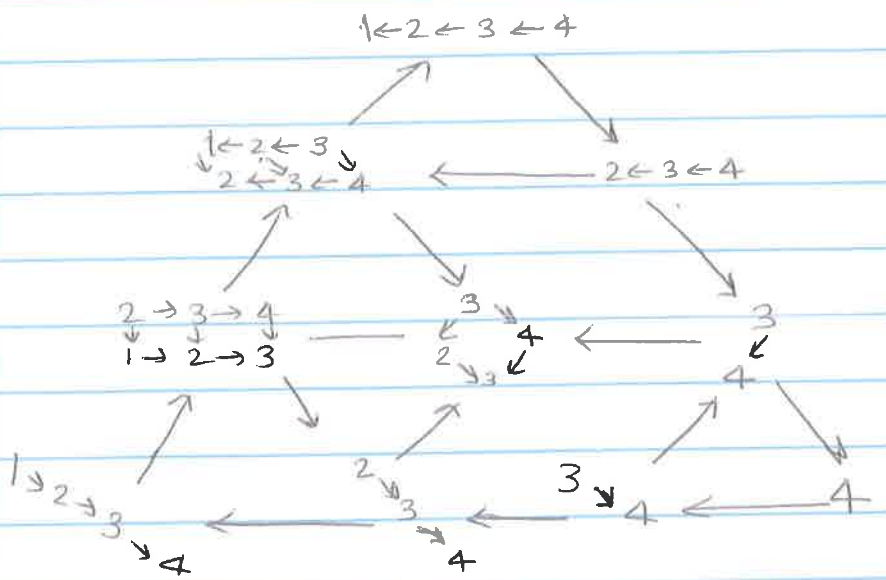
Say $T \in$ ~~A~~ Λ-model is maximally rigid if

$\mathrm{Ext}^1_\Lambda(T,T) = 0$ and $\mathrm{Ext}^1_\Lambda(T \oplus T', T \oplus T') = 0$

$\Rightarrow T' \in \mathrm{add}(T)$

each maximal rigid Λ-module has

$r = (\#$ positive root$)$ summands



$V = V_1 \oplus \cdots \oplus V_r^i$

Dualizing Lusztig's obtain for each $x \subset \Lambda$-mod

a regular function $\varphi_x \in \mathbb{C}[N]$

$$\varphi_x(x_{i_1}(t_1) \cdots x_{i_r}(t_r)) = \sum_{\underline{a} \in \mathbb{N}^r} \chi(\mathcal{F}_{\underline{i},\hat{\underline{a}}}(x))\, t_1^{a_1} \cdots t_r^{a_r}$$

$$\mathcal{F}^1_{\underline{i},a}(x) = \{ 0 = X_0 = X_1 \subset \cdots \subset X_r = X \mid X_{j+1}/X_j \cong S_{i_j}^{a_j},\ j = 1, \dots, r \}$$

$$x_i(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & \ddots & 1 \end{pmatrix} \leftarrow i$$

If x is in general position in some irred op φ of $\Lambda_{\underline{d}}$ then $\varphi_x \in \varphi^*$

in parti, if x is rigid $\Rightarrow \varphi_x \in \varphi^*$

If T $\overset{= T_1 \oplus \cdots \oplus T_k}{}$ is max. rigid & Γ_T : quiver of $\mathrm{End}_\Lambda(T)$

then :

- If T_k is not proj inj $\Rightarrow \exists !$ T_k' indep s.t.

  $\mu_k(T/T_k \oplus T_k^*)$ is max rigid

  $\exists$ $0 \to T_k \to \bigoplus T_{ta} \to T_k' \to 0$

  $\Gamma_{\mu_k(T)} \cong \mu_k(\Gamma_T)$

  $$\varphi_{T_k} \cdot \varphi_{T_k'} = \prod_{\substack{a \in \Gamma_T \\ sa = k}} \varphi_{X_{T_k}} + \prod_{\substack{b \in \Gamma_T \\ tb = k}} \varphi_{T_{sb}}$$

- $\exists$ rigid modules $M_1, \dots, M_r$ obtained from V by mutation

  $\mathbb{C}[N] = \mathbb{C}[\varphi_{M_1}, \dots, \varphi_{M_r}]$

  $\Rightarrow \mathbb{C}[N]$ cluster alg with $\{$cluster monomials$\} \subset \varphi^*$

3. Interpretation of φ^* in terms of QP's.

Formula : $\forall$ T max. rigid and reachable from V (by seq. of mutations)

$$\varphi_x = \varphi_T^{\underline{\dim}\, \mathrm{Hom}_\Lambda(T,x)} \cdot B_T \sum_{\underline{d} \in \mathbb{N}^m} \chi\left(\mathrm{Gr}_{\underline{d}}^{\mathcal{E}_T}(\mathrm{Ext}^1_\Lambda(T,x))\right) \hat{\varphi}_T^{\underline{d}} \qquad (*)$$

$\mathcal{E}_T = \mathrm{End}_\Lambda(T)^{gr}$ is a Jacobi alg

$B_T \longleftrightarrow \Gamma_T$

key observation : $\mathcal{F}\ell^{\Lambda}_{\underline{i}, a}(x) \cong \mathrm{Gr}^{\mathcal{E}(V)}_{\underline{d}(i,a)}(\mathrm{Ext}^1_\Lambda(\Omega V, x))$

In this laguage φ^* is given by functions of the form (*) with evaluated by at $\mathcal{E}_T$-modules M which are in general positioned of irred component of $\mathrm{rep}_{\mathcal{E}_T}(d)$ which is strongly reduced.
(φ is strongly reduced if $\varphi' \overset{\text{open}}{\subset} \varphi$, $\varphi' \ni M$ with some condition on codim & dim of $\mathrm{Hom}(T, TM)$)