Model structures for $(\infty, 1)$-categories

Julie Bergner

University of Virginia

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Idea of $(\infty, 1)$-categories

There are two ways to think conceptually about $(\infty, 1)$-categories.

One is homotopy-theoretic: An $(\infty, 1)$-category is the data of a homotopy theory, usually thought of as a category with some choice of weak equivalences. (Think of topological spaces and weak homotopy equivalences.)

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One is homotopy-theoretic: An $(\infty, 1)$-category is the data of a homotopy theory, usually thought of as a category with some choice of weak equivalences. (Think of topological spaces and weak homotopy equivalences.)

The other is more categorical: an $(\infty, 1)$-category is some kind of higher categorical structure which models the structure of a category "up to homotopy".

Why the name?

A category has objects and morphisms between objects, and those morphisms are required to compose when appropriate, and that composition is associative. Objects also have identity morphisms.

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If we have n-morphisms for arbitrarily large n, then we have an ∞-category.

In an $(\infty, 1)$-category, the n-morphisms are (weakly) invertible for all $n > 1$.

Taking a step back: $(\infty, 0)$-categories

How can we think about such a structure concretely?

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A general principle is that ∞-groupoids should just be topological spaces.

Points in a space are objects, paths are 1-morphisms, homotopies are 2-morphisms, and so forth. But paths and homotopies are invertible up to homotopy.

$(\infty, 1)$-categories as enriched categories

Another general principle for higher categories is that an n-category should be a category enriched in $(n-1)$-categories, perhaps in some weak sense. In other words, an the morphisms between two objects in an n-category should be equipped with the structure of an $(n-1)$-category.

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Similarly, an $(\infty, 1)$-category should be a category enriched in $(\infty, 0)$-categories, or a category enriched in spaces.

We could extend to more general (∞, n)-categories as categories enriched in $(\infty, n-1)$-categories.

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But, for many examples we'd like models which are not so strict, for example for which composition is only defined up to homotopy.

Model categories for $(\infty, 1)$-categories

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Recall that a model category is a category together with a choice of weak equivalences, as well as other distinguished morphisms called fibrations and cofibrations, satisfying some axioms.

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Recall that a model category is a category together with a choice of weak equivalences, as well as other distinguished morphisms called fibrations and cofibrations, satisfying some axioms.

The idea is that, between objects which are "fibrant" and "cofibrant", one can take homotopy classes of maps and obtain a good homotopy category.

Sometimes we have explicit descriptions of these nice objects, but not always.

Other properties of model categories

Model categories can also have additional structures.

- A model category is *left proper* if pushouts of weak equivalences along cofibrations are weak equivalences.
- It is *right proper* if pullbacks of weak equivalences along fibrations are weak equivalences.
- It is *proper* if it is both left and right proper.

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- It is *right proper* if pullbacks of weak equivalences along fibrations are weak equivalences.
- It is *proper* if it is both left and right proper.
- A model category is *simplicial* if the underlying category is enriched in simplicial sets, in a way compatible with the model structure.
- It is *cartesian* if it is enriched in itself, again in a compatible way.

Weak equivalences for simplicial categories

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The following definition is a natural extension of the definition of equivalence of categories.

Definition

A functor $f : \mathcal{C} \to \mathcal{D}$ between simplicial categories is a *Dwyer-Kan equivalence* if

- for any $x, y \in \text{ob}(\mathcal{C})$, $\text{Map}_{\mathcal{C}}(x, y) \to \text{Map}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets, and
- the functor $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

The model structure for simplicial categories

Theorem (B.)

There is a model structure on the category of small simplicial categories in which the weak equivalences are the Dwyer-Kan equivalences.

This model structure is proper but neither simplicial nor cartesian.

Both its fibrant and cofibrant objects are well-understood.

Simplicial spaces

Now we want to consider models with composition which is not defined so strictly.

As a foundation, we work in the setting of simplical spaces.

Definition

A *simplicial space* is a bisimplicial set, or functor $\mathbf{\Delta}^{op} \to \mathcal{SS}ets$.

The category of simplicial spaces can be given the Reedy model structure, whose weak equivalences are levelwise weak equivalences of simplicial sets.

The Segal condition

Given any simplicial space X, there are associated *Segal maps*

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}.$$

If X is the nerve of a simplicial category, then these maps are all isomorphisms. We relax this condition.

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Definition

A *Segal space* is a Reedy fibrant simplicial space such that the Segal maps for $n \geq 2$ are all weak equivalences of simplicial sets.

Thus Segal spaces have a kind of weak composition law.

Discretizing objects

However, a Segal space in some sense has a space, rather than a set, of objects.

One way to remedy this issue is simply to require the space in degree zero to be discrete.

Definition

A *Segal category* is a Segal space X such that X_0 is discrete.

A model structure for Segal categories

Theorem (Pelissier, B.)

There is a model structure on the category of simplicial spaces with discrete 0-space such that the fibrant objects are Segal categories. The weak equivalences are analogous to Dwyer-Kan equivalences of simplicial categories, and all objects are cofibrant.

This model structure is left proper, simplicial, and cartesian.

Another model structure for Segal categories

For technical reasons, it is nice to have another model structure for Segal categories with the same weak equivalences.

Theorem (B.)

There is an analogous model structure in which the fibrant objects are projective, rather than Reedy, fibrant.

In this model structure, not all objects are cofibrant, but it is still left proper. It is still simplicial, but it is not cartesian.

The completeness condition

However, actual discreteness is a difficult requirement when doing homotopy theory. An alternative is given by complete Segal spaces.

In a Segal space X, we can think of X_1 as the "morphisms" of X.

There is also a subspace of homotopy equivalences $X_{heq} \subseteq X_1$.

The image of the degeneracy map $s_0 \colon X_0 \to X_1$ lies in X_{heq}.

Definition

A Segal space is *complete* if this map $X_0 \to X_{heq}$ is a weak equivalence of simplicial sets.

The model structure for complete Segal spaces

Theorem (Rezk)

There is a model structure on the category of simplicial spaces in which the fibrant objects are the complete Segal spaces. It is obtained as a localization of the Reedy model structure.

This model structure is left proper, simplicial, and cartesian. All its objects are cofibrant.

Quasi-categories

Another model is given in the context of simplicial sets.

Recall that we have the following simplicial sets:

- the n-simplex $\Delta[n]$;
- its boundary $\partial\Delta[n]$; and
- for each $0 \leq k \leq n$, the k-horn $V[n, k]$, given by removing the edge opposite the kth vertex of $\partial\Delta[n]$.

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Definition

A *quasi-category* is a simplicial set K such that a lift exists in any diagram

$$\begin{array}{ccc} V[n,k] & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

where $n \geq 2$ and $0 < k < n$.

The model structure for quasi-categories

Theorem (Joyal, Lurie, Dugger-Spivak)

There is a model structure on the category of simplicial sets in which the fibrant objects are the quasi-categories. Again, the weak equivalences are analogous to Dwyer-Kan equivalences.

This model structure is left proper and cartesian. All its objects are cofibrant. It is not simplicial.

Relative categories

Returning to the homotopy-theoretic motivation for $(\infty, 1)$-categories, we can also simply think of categories equipped with a choice of weak equivalences, also called *relative categories*.

Theorem (Barwick-Kan)

There is a model structure on the category of relative categories. Weak equivalences and fibrations are described via a nerve-type functor to the complete Segal space model structure.

This model structure is known to be left proper, and we can describe the cofibrant objects. We don't know much about other properties, however.

What will we be doing?

The purpose of our team's investigations will be to look at some of these model structures and their properties.

Relative categories and the second model structure for Segal categories, have been investigated much less than the others, and some of their properties are either unknown or not well-documented in the literature.

In other cases, proofs of certain properties are consequences of more abstract results, for example that the model category for simplicial categories is left proper.

We'd like to give explicit proofs of these properties, establish ones that are known, and give explicit counterexamples for properties which do not hold.