# $K$-theory and fixed point invariants

Notes from a talk by: Cary Malkiewich and Kate Ponto
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This is about some new ideas about how to think about fixed points. This has lots of applications, which we want to advertise. I'll start with some historical context, and then Kate will tell you what the new ideas are.

Suppose you have a compact manifold $M$ (but a lot of this goes through for any finite cell complex) and a map $f : M \to M$.

**Question 1.** Does $f$ have a fixed point? If so, can I modify $f$ by a homotopy so as to remove those fixed points?

Basically we're asking whether $f$ has "essential" fixed points, that can't be removed. This is a classical question.

**Theorem 2** (Brouwer)**.** *If $M \to D^n$ then $f$ has an essential fixed point.*

**Theorem 3** (Lefschetz)**.** *Let $L(f) = \sum_i (-1)^i \operatorname{tr}(f; H_i(M;\mathbb{Q}))$ (i.e. $f$ acts on $H_i(M;\mathbb{Q})$ and you take the trace of the resulting matrix). If $M$ is any compact manifold or finite cell complex, then*

$$L(f) = \sum_{x \in \operatorname{Fix}(f)} \operatorname{ind}(x).$$

The idea is that $L(f)$ is defined to be a global thing, and the theorem relates it to a local property around fixed points. If there are no fixed points, then $L(f) = 0$.

**Corollary 4.** *If $L(f) \neq 0$ then $f$ has a fixed point.*

But we don't know the converse—maybe there were multiple fixed points that cancel out.

Wecken worked with the *Reidemeister trace* $R(f)$, which you should think of as an upgrade of $L(f) \in \mathbb{Z}$ to the Hochschild homology $HH_0$ of some ring—you should think of this as *more information* than $L(f)$.

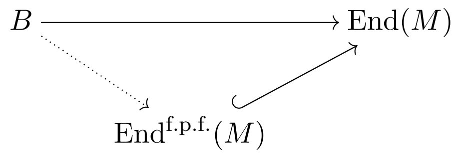
**Theorem 5** (Wecken)**.** *$R(f) = 0$ iff $f$ is homotopic to something fixed point free.*

So this is a complete obstruction!

There have been many generalizations.

- Suppose $G$ acts on $M$ and $f$ is equivariant w.r.t. $G$, and suppose you want the homotopy to respect this.

- Suppose you ave a family of maps, i.e. a map $B \to \mathrm{End}(M)$, and you want to continuously deform $f$ to it removes all the fixed points, i.e. you want a factorization




For $B = I$, in the 90's Geoghegan-Nicas constructed an invariant $R_B(f) \in HH_1$(same ring). So you add a 1-parameter family, and your invariant goes up one degree in $HH_*$. But this was rather hard to define. Idea: replace $HH$ with $THH$, and $R(f)$ is picking up on an element in $\pi_0 THH$, whereas $R_B(f)$ is picking up on an element in $\pi_1 THH$.

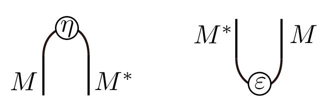
**Main problem 6.** Find the right conceptual framework for $R(f)$.

You want something like $L(f)$ that ends up being a sum over fixed points, and you want to relate that to something more global.

Kate solved this problem.

To make this good you want to be able to talk about $THH$ cleanly. In order to explain this, I need to go back to the 70's. Dold had the observation that these local-to-global results should just be functoriality. He constructed a generalization of the trace of a matrix: he defined a trace in any symmetric monoidal category. Think about vector spaces or modules over a commutative ring.

I'm going to draw these as string diagrams: | represents a module $M$ and | | represents the tensor product of two of them. In linear algebra, you can only define the trace over finite-dimensional vector spaces. In particular, I need unit and counit morphisms $k \to M \otimes M^*$ and $M^* \otimes M \to k$. Draw these as follows:

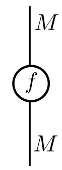



How to read these diagrams: e.g. in the first one (representing the unit) the empty space above the diagram represents the unit, the curved line represents the morphism $\eta$, and | | represents $M \otimes M^*$)

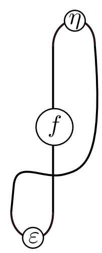
Bent strings can be straightened. The following composites are the identity.

In the context of vector spaces, this is exactly saying that your vector space is finite-dimensional; in the context of modules it's saying your module is finitely-generated and projective.

Now we can define the trace of an endomorphism $f : M \to M$




by feeding the input into the output:




The place where one line passes under the other is using the symmetric monoidal symmetry isomorphism: $k \xrightarrow{\eta} M \otimes M^* \xrightarrow{\tau} M^* \otimes M \xrightarrow{\varepsilon} k$.

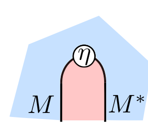
This is invariant under cyclic permutation (this is what trace "should" do).

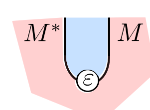
If $F$ is a strong symmetric monoidal functor, then

$$F(\mathrm{tr}(f)) = \mathrm{tr}(F(r)).$$

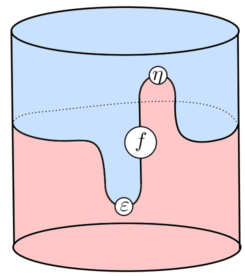
This is a diagram chase. If $F = H_*(-;\mathbb{Q})$ then this is the Lefschetz fixed point theorem. (This is due to Dold-Puppe.) The index (local phenomenon) comes from the trace in the stable homotopy category. The trace in chain complexes is the alternating sum of levelwise traces.

Issue: the Reidemeister trace doesn't fit in a symmetric monoidal category. Solution: add one more level of "stuff" by replacing the symmetric monoidal category by a bicategory. If we think of our symmetric monoidal category as $R$-modules, then there is an associated bicategory that has rings, bimodules, and their homomorphisms. How to relate that to string diagrams? Color the regions:




($M$ has red on the right side and blue on the left side, which means $M^*$ has blue on the right side and red on the left side.) This works for the identity morphisms as well, but you run into problems with the trace: you can't coherently assign colors to this one. To fix this problem, draw these pictures on a cylinder.




To have a trace, the bicategory must have more structure... like $HH$ and $THH$. The defining property here is that you can cycle things around and you don't change anything. (The idea of the cylinder is that you can just rotate your object and nothing changes.) The new invariant respects this new structure.

The Reidemeister trace is an invariant that is additive and $THH$-valued. This should set of alarm bells in your head—you should think about algebraic $K$-theory. Look at $TR$ and $TC$, and look for a class in $K$-theory that maps down to the Reidemeister trace. This is what we'll look at in the next day and a half.

**Theorem 7** (Malkiewich, P.)**.** *There is a class in $TR$ that lifts $R(f)$ and recovers information about $f^n$.*

**Theorem 8** (Campbell-P.)**.** *$R(f)$ is the expected image of a class in $K(-)$.*

This is just a start. For example, we want to revisit this with bundles. There should be compatibilities for traces associated to different iterates. We know this for $L(f)$ and want to do this for $R(f)$.