Foundations of $(\infty, 2)$-category theory

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The aim of this project is to develop some usable and comprehensible foundations of 2-category theory. Like $(\infty, 1)$-categories, these are "infinite-dimensional" categories where there are 2-morphisms between the morphisms, 3-morphisms between the 2-morphisms, etc., but at level 3 and higher, things are weakly invertible.

This has applications. For example, Ayala, Mazel-Gee, and Rozenblyum study the $(\infty, 1)$-category of genuine G-spectra (for G a Lie group) and they describe a universal property of an object in $(\infty, 2)$-categories.

Aim: prove theorems transfer across models—you want to apply theorems proven for one model to examples constructed in a different model.

In this talk, I'll explain how this was achieved for $(\infty, 1)$-categories.

Step 0: develop models of $(\infty, 1)$-categories. One model is quasi-categories (weak Kan complexes), first defined by Boardman-Vogt and the homotopy theory is developed by Joyal. Another model is Segal categories, defined by Hirschowitz-Simpson, and the homotopy theory is worked out by Pellissier and Bergner. There's also complete Segal spaces (Rezk spaces) due to Rezk. There are naturally marked quasi-categories, defined by Roberts-Street and the homotopy theory is due to Verity and Lurie.

Step 1: develop the analytic theory of $(\infty, 1)$-categories in a particular model.

Definition 1 (Joyal)**.** A *terminal object* in a quasi-category A is a map $\Delta^0 \xrightarrow{a} A$ such that for any $\partial\Delta^n \to A$ such that $\Delta^0 \xrightarrow{n} \partial\Delta^n \to A$ is a, there is a filler

$$\begin{array}{ccccc} \Delta^0 & \longrightarrow & \partial\Delta^n & \longrightarrow & A \\ & & \downarrow & \nearrow & \\ & & \Delta^n & & \end{array}$$

(We're calling this "analytic" because it's quasi-category-specific.) A *limit* of a diagram $J \xrightarrow{j} A$ is a terminal object in the category of cones, defined as the pullback

$$\begin{array}{ccc} \mathrm{Cones}/j & \longrightarrow & A^{\Delta^0 * J} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{j} & A^I \end{array}$$

Definition 2 (Lurie)**.** An *adjunction* between quasi-categories is a quasi-category over Δ^1, i.e. $M \to \Delta^1$, such that $A \simeq M_1$ and $B \simeq M_0$ (i.e. the fibers over 1 and 0) that is cocartesian (represented by a functor $u : A \to B$) and cartesian (represented contravariantly by a functor $f : B \to A$).

Step 2: develop the synthetic theory of $(\infty, 1)$-categories. So we've defined all these things for quasi-categories, and we'd like to think about this in a model-independent way.

What do various models have in common? Think about a model category that has these as objects.

Theorem 3. *Quasi-categories, Segal categories, complete Segal spaces, and naturally marked quasi-categories are the fibrant-cofibrant objects in model categories enriched over* $s\,\mathrm{Set}$ *with the Joyal model structure.*

E.g. in the Segal categories model, if A and B are Segal categories, then $\mathrm{Fun}(A, B)$ can be taken to be a quasi-category.

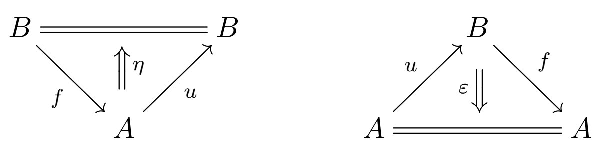
Corollary 4 (Riehl-Verity)**.** *For each model of* $(\infty, 1)$*-categories, there exists a strict 2-category where*

- *the objects are the* $(\infty, 1)$*-categories*
- $\hom(A, B) := \mathrm{ho}\,\mathrm{Fun}(A, B)$*. This means that the 1-morphisms* $A \to B$ *are morphisms in the model*
- *a 2-morphism* α *between* $f, g : A \to B$ *is a homotopy class* $f \xrightarrow{\alpha} g \in \mathrm{Fun}(A, B)$

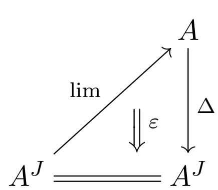
I'm going to redefine these analytic notions synthetically. The first definition works in any 2-category.

Definition 5. An *adjunction* is:

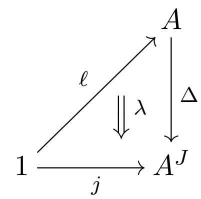
- a pair A, B
- $u : A \to B$, $f : B \to A$
- unit and counit 2-morphisms:



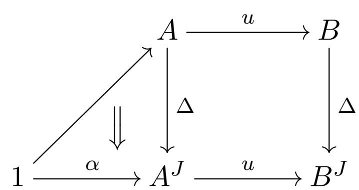
Definition 6 (Riehl-Verity)**.** A *terminal object in* A is a right adjoint to the unique map $A \to 1$. A *limit functor* of shape J is a right adjoint to $\Delta : A \to A^J$. Equivalently, there is an *absolute lifting diagram* (dual to the property of being an absolute Kan extension)



"Absolute" means that the universal property is stable under mapping to the left-most A^J. The limit of $1 \xrightarrow{j} A^J$ is an absolute lifting diagram



Proposition 7 (RV)**.** *Right adjoints preserve limits. There is an absolute diagram*

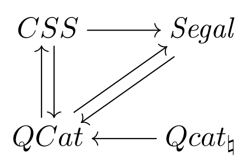


Question 8. Do analytic theorems transfer across models?

Step 3: prove model independence.

People have developed comparisons between models (e.g. Julie Bergner's work). The comparisons that will be most useful to us are:

Theorem 9 (Joyal-Tierney, Bergner)**.** *There exist right Quillen equivalences compatible with Joyal*



Corollary 10 (RV)**.** *The 2-categories are* biequivalences*—essentially surjective on objects up to equivalence, and local equivalence on homs.*

So you get a lot of bijections between things.

A consequence of having these biequivalences is:

Theorem 11 (RV)**.** *A change of model functor preserves, reflects, and creates all the category theory of* $(\infty, 1)$*-categories.*

There are things we only know how to prove in particular models, e.g. left and right Kan extensions for complete and cocomplete $(\infty, 1)$-categories. But now we have this for all models.

I'd like to have this for $(\infty, 2)$-categories.