Partitions into Values of a Polynomial

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- We can study other partition functions by restricting the set of allowable parts:
  Partitions into odd numbers, squares, primes, etc.

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- There are similarities to Waring's problem, but we are not restricting the number of parts.

Hardy & Ramanujan

Theorem (Hardy - Ramanujan, 1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

[Notation: $f(x) \sim g(x)$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$]

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Claim (Hardy and Ramanujan, 1918)

$$\log p^k(n) \sim (k+1)\left(\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)} n^{1/(k+1)}.$$

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- In 2014, R.C. Vaughan revisited the problem, providing a much simpler proof for $p^2(n)$ with a decent error term.

Partitions into k-th powers

Theorem (G., 2016)

$$p^k(n) = \frac{\exp\left(\frac{k+1}{k^2}\zeta(\frac{k+1}{k})\Gamma(\frac{1}{k})X^{\frac{1}{k}} - \frac{1}{2}\right)}{(2\pi)^{\frac{k+2}{2}}X^{\frac{3}{2}}Y^{\frac{1}{2}}}\left(\pi^{\frac{1}{2}} + \sum_{h=1}^{2J}\frac{c_h}{Y^{\frac{h}{2}}} + O(\frac{1}{Y^J})\right)$$

Here X and Y satisfy $X \sim C_1 n^{k/(k+1)}$ and $Y \sim C_2 n^{1/(k+1)}$.

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Consequence

$$\log p^k(n) \sim (k+1)\left(\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)} n^{1/(k+1)}.$$

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Partitions into Polynomials

- Fix a polynomial $f(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$, and let $\mathcal{S}_f = \{f(m) : m \in \mathbb{N}\}$.

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  - f is integer-valued,
  - $\mathrm{GCD}(\mathcal{S}_f) = 1$,
  - $\Re(\alpha_j) < 1$ for each j.
- Goal: Find an asymptotic formula for $p_f(n)$.

Example

- Let $f(x) = \frac{1}{2}x(x+1)$. Then $\mathcal{S}_f = \{1, 3, 6, 10, \ldots\}$.
- Thus $p_f(n)$ counts the number of partitions into triangle numbers.
- $p_f(10) = 7$:

$$10, \quad 6+3+1, \quad 6+1+\cdots+1, \quad 3+3+3+1,$$

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Special Cases

- (Vaughan, 2014) $f(x) = x^2$. Partitions into squares

$$\log p^2(n) \sim 3\left(\frac{1}{4}\sqrt{\pi}\zeta\left(\frac{3}{2}\right)\right)^{2/3} n^{1/3}$$

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- (Berndt, Malik, Zaharescu, 2017) $f(x) = (bx+a)^k, \quad (a,b) = 1$. Partitions into powers of an arithmetic progression

$$\log p_{k;a,b}(n) \sim (k+1)\left(\frac{1}{bk}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{\frac{k}{k+1}} n^{\frac{1}{k+1}}.$$

General Polynomials

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Claim

The methods of Vaughan, G., and Berndt-Malik-Zaharescu can be extended to prove that

$$\log p_f(n) \sim (k+1)\left(\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)}\left(\frac{n}{a}\right)^{1/(k+1)}$$

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This implies that the number of partitions of n into triangle numbers $[f(x) = \frac{1}{2}x(x+1)]$ grows roughly like the number of partitions of $2n$ into squares $[f(x) = x^2]$.

Generating Functions

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We can use Cauchy's Theorem to extract the coefficient of z^n:

$$p_f(n) = \int_0^1 \rho^{-n} \Psi_f(\rho e^{2\pi i \Theta}) e^{-2\pi i n \Theta} \, d\Theta.$$

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and our problem is now

$$p_f(n) = \int_0^1 \rho^{-n} \exp(\Phi_f(\rho e(\Theta)) - 2\pi i n \Theta)\, d\Theta.$$

Hardy-Littlewood Circle Method

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The critical step is to consider objects of the form $\sum_{x \le n} e(\Theta f(x))$.

If Θ is "close" to a rational number with "small" denominator, then we can obtain "good" estimates for the integrand.
These regions make up the *Major Arcs*, and contribute to the main term.
The gaps between the major arcs are called *Minor Arcs*.

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We will split the integral into three parts:

$$\int_0^1 = \int_{\mathfrak{M}(1,0)} + \int_{\mathfrak{M}\setminus\mathfrak{M}(1,0)} + \int_{\mathfrak{m}}$$

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Main Term Heuristics

Let $f(x) = a(x-\alpha_1)\cdots(x-\alpha_k)$. We need to estimate

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The residue of $\sum_{n=1}^{\infty} f(n)^{-s}$ at $s = 1/k$ is

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- Recall: $f(x) = a(x-\alpha_1)\cdots(x-\alpha_k)$, residue of $\sum_{n=1}^{\infty} f(n)^{-s}$ at $s = 1/k$ is $\frac{1}{ka^{1/k}}$.

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- Powers: $f(x) = 1x^k$

$$\log p_f(n) \sim (k+1)\left(\frac{1}{1k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)} n^{1/(k+1)}.$$

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- The error term will be calculated using delicate Hardy-Littlewood method arguments.

Thank You!