Sums of distinct divisors

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The “anatomy” of integers

What does it mean to study the “anatomy” of integers?
Some natural problems/goals:

- Study the prime factors of integers, their size and their quantity.
- Obtain good bounds for the number of integers with certain properties (e.g., those with only large prime factors).
- Understand the distribution of divisors of integers in a given interval.
Several papers in this area study the set of integers with $z$-dense divisors.

Let $1 = d_1(n) < d_2(n) < \cdots < d_{\tau(n)}(n) = n$ denote the sequence of divisors of an integer $n$.

**Definition**

For $z \geq 2$, an integer $n$ is $z$-dense if

$$\max_{1 \leq i < \tau(n)} \frac{d_{i+1}(n)}{d_i(n)} \leq z.$$ 

**Example.** 6 is 2-dense, since $\frac{2}{1}, \frac{3}{2}, \frac{6}{3} \leq 2$. 
Integers with dense divisors

**Introduction**

**Practical numbers**

$\varphi$-practical numbers

**Asymptotics**

A generalization

**Integers with dense divisors**

$s_{\text{distinct divisors}}$

Lola Thompson

**Theorem (Tenenbaum)**

For $n \geq 2$,

$$F(n) := \begin{cases} 1 & n = 1 \\ \max \{dP^-(d) : d | n, d > 1\} & n \geq 2. \end{cases}$$

So $n$ has $z$-dense divisors if $F(n) \leq nz$. 

$$F(n) := \begin{cases} 1 & n = 1 \\ \max \{dP^-(d) : d | n, d > 1\} & n \geq 2. \end{cases}$$
Let \( D(x, z) = \#\{n \leq x : n \text{ is } z\text{-dense}\} \).

Tenenbaum obtained upper and lower bounds for \( D(x, z) \), which were later improved by Saias to

\[
D(x, z) \asymp \frac{x \log z}{\log x}.
\]
Proof sketch

For any fixed $z \geq 2$, let $D(x) = \{n \leq x : F(n) \leq nz\}$.

Let $D(x, y) = \#\{n \in D(x) : P^+(n) \leq y\}$.

Then

$$D(x, y) = 1 + \sum_{p \leq \min(y, h(x))} D(x/p, p) + ET.$$

(E.g., Can take $h(x) = \sqrt{x}$ so that ET is negligible.)

Smoothed version: $D^*(x, y) = \int_1^{\min(y, \sqrt{x})} D^*(x/t, t) \frac{dt}{\log t}$.

Can *almost* take $D^*(x, y) = x \rho(u - 1)/\log x$, where $\rho$ is defined by $u\rho'(u) + \rho(u - 1) = 0$ and $\rho(u) = 1$ for $u \leq 1$. 

Our talk will focus on two natural applications of Tenenbaum’s work on integers with $\omega$-dense divisors:

1. How often is it the case that every $m$ in $[1, n]$ can be written as a sum of distinct divisors of $n$?
Our talk will focus on two natural applications of Tenenbaum’s work on integers with \( z \)-dense divisors:

1. How often is it the case that every \( m \) in \([1, n]\) can be written as a sum of distinct divisors of \( n \)?

2. How often does the polynomial \( x^n - 1 \) have a divisor of every degree between 1 and \( n \) in \( \mathbb{Z}[x] \)?
Practical numbers

**Definition**

A positive integer \( n \) is **practical** if every \( m \) with \( 1 \leq m \leq \sigma(n) \) can be written as a sum of distinct divisors of \( n \).

**Example.** \( n = 6 \)

Divisors: 1, 2, 3, 6

**Nonexample.** \( n = 10 \)

Divisors: 1, 2, 5, 10
Practical numbers: a short history

Theorem (Erdős, 1950)

Let \( PR(X) := \# \{ n \leq X : n \text{ is practical} \} \). Then

\[ PR(X) = o(X). \]
Hausman and Shapiro, 1983:

\[ PR(X) \ll \frac{X}{(\log X)^\beta}, \quad \beta = \frac{(1 - 1/\log 2)^2}{2} = 0.0979.... \]

Margenstern, 1984:

\[ PR(X) \gg \frac{X}{\exp(\alpha(\log \log X)^2)}, \quad \alpha = \frac{1 + \varepsilon}{2 \log 2} = 0.7213.... \]

Tenenbaum, 1986:

\[ \frac{X}{(\log X)(\log \log X)^{4.21}} \ll PR(X) \ll \frac{X(\log \log X)(\log \log \log X)}{\log X}. \]
Practical numbers: a short history

Theorem (Saias, 1997)

For all $X \geq 2$,

$$PR(X) \leq \frac{X}{\log X}.$$
Definition

A positive integer $n$ is φ-practical if every $m$ with $1 \leq m \leq n$ can be written as $\sum_{d \in D} \varphi(d)$, where $D$ is a subset of divisors of $n$.

Note: Since $x^n - 1 = \prod_{d|n} \Phi_d(x)$, this is equivalent to the condition that $x^n - 1$ has at least one divisor of every degree between 1 and $n$. 
\(\varphi\)-practical example

**Example.**  \(n = 6\)
- Divisors: 1, 2, 3, 6
- \(\varphi\) values: 1, 1, 2, 2

**Nonexample.**  \(n = 66\) is practical but **not** \(\varphi\)-practical.
- Divisors: 1, 2, 3, 6, 11, 22, 33, 66
- \(\varphi\) values: 1, 1, 2, 2, 10, 10, 22, 22
\(\varphi\)-practical example

**Example.** \(n = 6\)

Divisors: 1, 2, 3, 6
\(\varphi\) values: 1, 1, 2, 2

**Nonexample.** \(n = 66\) is practical but **not** \(\varphi\)-practical.

Divisors: 1, 2, 3, 6, 11, 22, 33, 66
\(\varphi\) values: 1, 1, 2, 2, 10, 10, 22, 22

**Exercise.** Every even \(\varphi\)-practical is practical.
We can prove the following analogue of Saias’ result for the $\varphi$-practical numbers:

**Theorem (T., 2013)**

Let $F(X) = \# \{ n \leq X : n \text{ is } \varphi\text{-practical} \}$. Then

$$F(X) \approx \frac{X}{\log X}.$$
A key obstruction

The proofs of Saias et al. relied heavily on the following:

**Theorem (Stewart, 1954)**

Let \( n = p_1^{e_1} \cdots p_j^{e_j}, n > 1, \) with \( p_1 < p_2 < \cdots < p_j \) prime and \( e_i \geq 1 \) for \( i = 1, \ldots, j. \) Then \( n \) is practical iff for all \( i = 1, \ldots, j, \) \( p_i \leq \sigma(p_1^{e_1} \cdots p_{i-1}^{e_{i-1}}) + 1. \)

Unfortunately, there’s no simple method for building up \( \varphi \)-practical numbers from smaller ones.

**Example.** \( 3^2 \times 5 \times 17 \times 257 \times 65537 \times (2^{31} - 1) \) is \( \varphi \)-practical, but none of the numbers \( 3^2, 3^2 \times 5, 3^2 \times 5 \times 17, \) \( 3^2 \times 5 \times 17 \times 257, 3^2 \times 5 \times 17 \times 257 \times 65537 \) are \( \varphi \)-practical.
Proof of the upper bound

Instead, we devise the following workaround:

**Definition**

Let \( n = p_1^{e_1} \cdots p_k^{e_k} \). Let \( m_i = p_1^{e_1} \cdots p_i^{e_i} \). We define an integer \( n \) to be **weakly \( \varphi \)-practical** if the inequality \( p_{i+1} \leq m_i + 2 \) holds for all \( i \).

**Lemma**

Every \( \varphi \)-practical number is weakly \( \varphi \)-practical.

**Note**: The converse does **not** hold. For example, 45 is not \( \varphi \)-practical but it is weakly \( \varphi \)-practical.
Proof of the upper bound

To prove our theorem, we consider two cases:

- If \( n \) is even & \( \varphi \)-practical then \( p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1 \) for all \( i \geq 1 \). Hence, each \( m_i \) satisfies the inequality in Stewart’s Condition, so \( n \) is practical.
Proof of the upper bound

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- If $n$ is even & $\varphi$-practical then $p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1$ for all $i \geq 1$. Hence, each $m_i$ satisfies the inequality in Stewart’s Condition, so $n$ is practical.

- On the other hand, observe that for every $n \in (0, X]$, there is a unique $k$ such that $2^k n \in (X, 2X]$. Then, if $n$ is odd & $\varphi$-practical, $2^k n$ will be practical.
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- If $n$ is even & $\varphi$-practical then $p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1$ for all $i \geq 1$. Hence, each $m_i$ satisfies the inequality in Stewart’s Condition, so $n$ is practical.

- On the other hand, observe that for every $n \in (0, X]$, there is a unique $k$ such that $2^k n \in (X, 2X]$. Then, if $n$ is odd & $\varphi$-practical, $2^k n$ will be practical.

Thus, $F(X) \leq PR(2X) \ll \frac{X}{\log X}$, by Saias’ Theorem.
Lower Bound Proof Sketch

Saias obtains his lower bound by comparing the set of practical numbers with the set of integers with 2-dense divisors:

**Definition**

An integer $n$ is 2-dense if $\max_{1 \leq i \leq \tau(n)} \frac{d_{i+1}(n)}{d_i(n)} - 1 \leq 2$.

**Note:** All integers with 2-dense divisors are practical, but the same cannot be said about the $\varphi$-practical numbers. For example, $n = 66$ is 2-dense but it is not $\varphi$-practical.
Lower Bound Proof Sketch

We obtain our lower bound by comparing the set of $\varphi$-practical numbers with the set of integers with strictly 2-dense divisors:

**Definition**

An integer $n$ is **strictly 2-dense** if

$$\max_{1 < i < \tau(n)} \frac{d_{i+1}(n)}{d_i(n)} < 2$$

and

$$\frac{d_2(n)}{d_1(n)} = 2 = \frac{d_{\tau(n)}(n)}{d_{\tau(n)-1}(n)}.$$

It turns out that all strictly 2-dense integers are $\varphi$-practical.
Lower Bound Proof Sketch

**Goal:** Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.
Lower Bound Proof Sketch

**Goal:** Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.

1. First find an upper bound for the number of integers up to $X$ that are 2-dense but not strictly 2-dense:

\[
\sum_{k>C} \sum_{m \in (2^k-1, 2^k)} \sum_{p \in (2^{k-1}, 2^{k+1})} \sum_{j \leq X/mp} \frac{mpj}{P^-(j)}>1.
\]
Goal: Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.

1. First find an upper bound for the number of integers up to $X$ that are 2-dense but not strictly 2-dense:

$$\sum_{k>C} \sum_{m \in (2^{k-1}, 2^k)} \sum_{p \in (2^{k-1}, 2^{k+1})} \sum_{j \leq X/mp} \frac{1}{mpj}$$

2. Use Brun’s sieve and other classical techniques from multiplicative number theory to show that the number counted above is $\leq \varepsilon \frac{X}{\log X}$. 
Lower Bound Proof Sketch

- Show that a subset of the strictly $2$-dense integers is in one-to-one correspondence with a positive proportion of the $2$-dense integers with obstructions at $k < C$.

**Corollary (T., 2013)**

For $X$ sufficiently large, we have

\[
\#\{n \leq X : n \text{ is practical but not } \varphi\text{-practical}\} \gg \frac{X}{\log X}.
\]

Moreover, we also have

\[
\#\{n \leq X : n \text{ is } \varphi\text{-practical but not practical}\} \gg \frac{X}{\log X}.
\]
An asymptotic for the practicals

Theorem (Weingartner, 2015)

There exists a positive constant \( c \) such that for \( X \geq 3 \),

\[
PR(X) = \frac{cX}{\log X} \left( 1 + O \left( \frac{\log \log X}{\log X} \right) \right).
\]
Proof sketch

**Idea:** Count all integers \( n \leq x \) according to their practical part:

\[
n = (p_1^{e_1} \cdots p_j^{e_j})(p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}) = m \cdot r
\]

where \( m = p_1^{e_1} \cdots p_j^{e_j} \) is **practical** but \( p_1^{e_1} \cdots p_j^{e_j} p_{j+1}^{e_{j+1}} \) is **not**.
Proof sketch

Idea: Count all integers $n \leq x$ according to their practical part:

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where $m = p_1^{e_1} \cdots p_j^{e_j}$ is practical but $p_1^{e_1} \cdots p_j^{e_j} p_j^{e_j+1}$ is not.

Then

$$\lfloor x \rfloor = \sum_{\substack{m \leq x \\ m \text{ practical}}} \Phi(x/m, \sigma(m) + 1)$$

where $\Phi(x, y) = \#\{n \leq x : p \mid n \Rightarrow p > y\}$. 
Proof sketch

Since

\[ [x] = \sum_{m \leq x} \Phi(x/m, \sigma(m) + 1) \]
Proof sketch

Since

\[ \left\lfloor x \right\rfloor = \sum_{m \leq x, \, m \text{ practical}} \Phi\left(\frac{x}{m}, \sigma(m) + 1\right) \]

\[ 1 = \sum_{m \text{ practical}} \frac{1}{m} \prod_{p \leq \sigma(m) + 1} \left(1 - \frac{1}{p}\right) \]
Proof sketch

Since

\[ \lfloor x \rfloor = \sum_{m \leq x, \ m \text{ practical}} \Phi\left(\frac{x}{m}, \sigma(m) + 1\right) \]

\[ 1 = \sum_{m \text{ practical}} \frac{1}{m} \prod_{p \leq \sigma(m) + 1} \left(1 - \frac{1}{p}\right) \]

we have

\[ 0 = \sum_{m \text{ practical}} \left( \Phi\left(\frac{x}{m}, \sigma(m) + 1\right) - \frac{\lfloor x \rfloor}{m} \prod_{p \leq \sigma(m) + 1} \left(1 - \frac{1}{p}\right) \right) \]
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Proof sketch

\[ 0 = \sum_{m \text{ practical}} \left( \Phi\left( \frac{x}{m}, \sigma(m) + 1 \right) - \frac{\left\lfloor \frac{x}{m} \right\rfloor}{m} \prod_{p \leq \sigma(m)+1} \left( 1 - \frac{1}{p} \right) \right) \]

Observe that:

\[ \Phi(x,y) \approx e^{\gamma} x \omega \left( \log x \log y \right) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \]

\[ \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \approx e^{-\gamma} \log y \]

\[ \log \sigma(m) + 1 \approx \log(2^m) \]

Use partial summation, get an integral equation, apply a Laplace transform, and invert the Laplace transform.
Proof sketch

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Proof sketch

\[0 = \sum_{m \text{ practical}} \left( \Phi\left(\frac{x}{m}, \sigma(m) + 1\right) - \frac{\lfloor x \rfloor}{m} \prod_{p \leq \sigma(m)+1} \left(1 - \frac{1}{p}\right) \right)\]

Observe that:

- \(\Phi(x, y) \approx e^\gamma x \omega \left(\frac{\log x}{\log y}\right) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)\)

- \(\prod_{p \leq y} \left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log y}\)

- \(\log(\sigma(m) + 1) \approx \log(2m)\)

Use partial summation, get an integral equation, apply a Laplace transform, and invert the Laplace transform.
An asymptotic for the \( \varphi \)-practicals

**Theorem (Pomerance, T., Weingartner, 2016)**

There exists a positive constant \( C \) such that for \( X \geq 2 \),

\[
F(X) = \frac{CX}{\log X} \left( 1 + O \left( \frac{1}{\log X} \right) \right).
\]
Proof Sketch: Starters

**Definition**

A **starter** is a $\varphi$-practical number $m$ such that either $m/P^+(m)$ is not $\varphi$-practical or $P^+(m)^2 \mid m$.

**Note:** A $\varphi$-practical number $n$ is said to have starter $m$ if $m$ is a starter, $m$ is an initial divisor of $n$, and $n/m$ is squarefree.

**Examples:**

- 4 is the only starter with squarefull part 4.
- There are only 3 starters with squarefull part 49: $294 = 2 \cdot 3 \cdot 7^2$, $1470 = 2 \cdot 3 \cdot 5 \cdot 7^2$, $735 = 3 \cdot 5 \cdot 7^2$.
- There are infinitely many starters with squarefull part 9.
Proof Sketch:

1. Partition $\varphi$-practicals according to their starters: $n = mb$.

2. Use Weingartner’s machinery to show that, for any fixed $m \geq 1$, there exist sequences of real numbers $c_m$ and $r_m$ such that

$$B_m(x) = c_m \frac{x}{\log x} + O \left( r_m \frac{x}{\log^2 x} \right),$$

where $B_m := \#\{\varphi$-practical $n \leq x$ with starter $m\}$.

3. Show that $\sum c_m$ and $\sum r_m$ are finite.
Estimating the asymptotic constant

We can use Sage to compute \( F(X)/\frac{X}{\log X} \):

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<th>( X )</th>
<th>( F(X) )</th>
<th>( F(X)/(X/\log X) )</th>
<th>( F(X)/(\text{Li}(X)) )</th>
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<td>0.989834</td>
<td>0.948988</td>
</tr>
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</table>

Table: Ratios for \( \varphi \)-practicals
A generalization

Let

\[ S_f(n) = \sum_{d|n} f(d) = f \ast 1(n). \]

**Definition**

A positive integer \( n \) is called \( f \)-**practical** if for every positive integer \( m \leq S_f(n) \) there is a set \( D \) of divisors of \( n \) for which

\[ m = \sum_{d \in D} f(d) \]

holds.

**Example** \( f = I \): practical numbers.

**Example** \( f = \varphi \): \( \varphi \)-practical numbers.
Example All positive integers are $\tau$-practical.
Densities of $f$-practicals

Example All positive integers are $\tau$-practical.

Example The set of $\lambda$-practical numbers has asymptotic density 0.
Densities of $f$-practicals

**Example** All positive integers are $\tau$-practical.

**Example** The set of $\lambda$-practical numbers has asymptotic density 0.

**Example** Let $g : \mathbb{N} \to \mathbb{N}$, where $g(1) = 1$, $g(2^k) = 2$, and $g(p^k) = 3$ for all $p \geq 3$ and all $k \geq 1$. The set of $g$-practical numbers has asymptotic density $1/2$. 
Densities of $f$-practicals

Theorem (Schwab, T., 2016)

For each $n \in \mathbb{N}$, there is a function $f_n$ such that the asymptotic density of $f_n$-practical numbers in $\mathbb{N}$ is $1 - \frac{\varphi(n)}{n}$. 

Corollary (Schwab, T., 2016)
The densities of $f$-practical sets are dense in $[0, 1]$. 

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Densities of $f$-practicals

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Other results

We also:

- Classified the multiplicative functions $f$ for which the $f$-practical numbers can be completely determined via a Stewart-like criterion;
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Other results

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- Classified the multiplicative functions $f$ for which the $f$-practical numbers can be completely determined via a Stewart-like criterion;

- Proved Chebyshev-type bounds for certain $f$-practical sets;

- Classified the additive functions $f$ for which all positive integers are $f$-practical.
Thank you!