The obstacle problem for the fractional Laplacian with drift

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Connections For Women: Harmonic Analysis

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In this talk I will overview this problem and present some new results on the regularity of the free boundary.

Classical obstacle problem

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Suppose we want to wrap a meatloaf in a plastic wrap.
Here the meatloaf is the obstacle, and the configuration of the plastic wrap, after it adjusts to the geometry of the meatloaf, represents the solution to the obstacle problem.

Formulation of the classical obstacle problem

We are given:

- $\phi \in C^2(D)$, the *obstacle*;
- $\psi \in W^{1,2}(D)$ with $\phi \leq \psi$ on ∂D, the *boundary values*;
- $f \in L^\infty(D)$, the *source term*.

We want to minimize

$$\int_D (|\nabla u|^2 + 2fu)dx$$

over $\mathcal{K} = \{u \in W^{1,2}(D) \ : u = \psi \text{ on } \partial D, u \geq \phi \text{ a.e. in } D\}$.

Classical obstacle problem

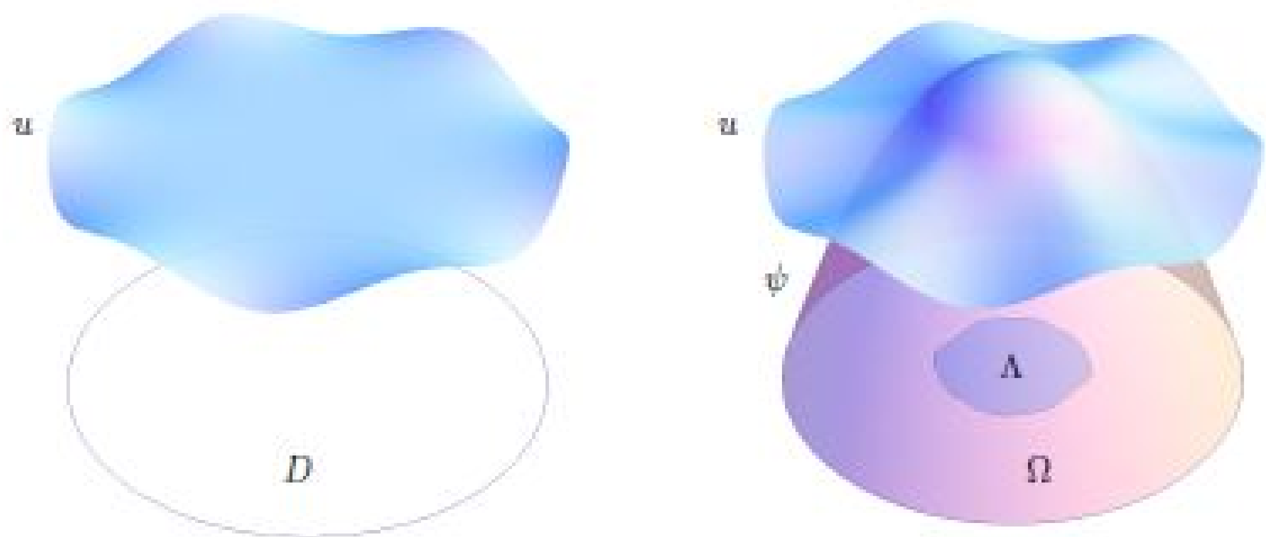

Figure: "Regularity of Free Boundaries in Obstacle-Type Problems", by Petrosyan, Shahgholian, Uraltseva

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$$\Delta u = f \text{ in } \{u > \phi\}$$

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- Coincidence set: $\Lambda_\phi(u) = \{x \in D \mid u(x) = \phi(x)\}$.
- Free boundary: $\Gamma_\phi(u) = \partial\{x \in D \mid u(x) = \phi(x)\}$.

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First fundamental question: How smooth is the solution? The optimal regularity of the solution is $u \in C^{1,1}_{\text{loc}}(D) \cong W^{2,\infty}_{\text{loc}}(D)$.

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Second fundamental question: How smooth is the free boundary? In 1977 Kinderlherer and Nirenberg proved that, if the free boundary is a C^1 hypersurface, then it is C^ω (real analytic). In the same year Caffarelli developed his theory of the regularity of the free boundary and proved Lipschitz regularity, and then proved how to go from Lipschitz to $C^{1,\alpha}$.

Obstacle problem for the fractional Laplacian

We study the obstacle problem defined by the fractional Laplacian with gradient perturbation

$$\min\{L\widehat{u}(x), \widehat{u}(x) - \widehat{\varphi}(x)\} = 0, \quad \forall\, x \in \mathbb{R}^n, \tag{0.1}$$

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where we denote

$$L\psi(x) := (-\Delta)^s\, \psi(x) + \langle b(x), \nabla\psi(x)\rangle + c(x)\psi(x), \quad \forall\, \psi \in C_0^2(\mathbb{R}^n).$$

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The action of $(-\Delta)^s$ on $\psi \in C_0^2(\mathbb{R}^n)$ is given by

$$(-\Delta)^s\psi(x) = c_{n,s}\ \text{p.v.} \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{|x-y|^{n+2s}}\, dy,$$

understood in the sense of the principal value.

Regularity of the solution

Assumptions:

- $s \in (1/2, 1)$,
- $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$,
- $c \in C^s(\mathbb{R}^n)$ with $c \geq 0$,
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- existence,
- uniqueness (assuming $c(x) \geq c_0 > 0, \ \forall x \in \mathbb{R}^n$, and $b \in C^{0,1}$),
- optimal regularity of solution, $\widehat{u} \in C^{1,s}(\mathbb{R}^n)$.

Regular set

Here our focus is the *free boundary*, defined as $\widehat{\Gamma}(\widehat{u}) := \partial\{\widehat{u} = \widehat{\varphi}\}$.

We will prove regularity of a special subset of $\widehat{\Gamma}(\widehat{u})$, the so called *regular set*, denoted by $\widehat{\Gamma}_{1+s}(\widehat{u})$.

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We will prove regularity of a special subset of $\widehat{\Gamma}(\widehat{u})$, the so called *regular set*, denoted by $\widehat{\Gamma}_{1+s}(\widehat{u})$.

The set of regular free boundary points, $\widehat{\Gamma}_{1+s}(\widehat{u})$, is, informally, the set of points of $\widehat{\Gamma}(\widehat{u})$ where the limit of a frequency function of Almgren type attains its smallest possible value - formal definition in a few slides :)

Main result: $C^{1,\gamma}$ regularity of the regular free boundary

Theorem (Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016)

Let $s \in (1/2, 1)$, $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$, $0 \leq c \in C^s(\mathbb{R}^n)$, $\widehat{\varphi} \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. Let $\widehat{u} \in C^{1,s}(\mathbb{R}^n)$ solve (0.1) and $x_0 \in \widehat{\Gamma}_{1+s}(\widehat{u})$. Then $\exists \gamma \in (0,1)$ and $\eta > 0$, such that

$$B_\eta(x_0) \cap \widehat{\Gamma}(\widehat{u}) \subseteq \widehat{\Gamma}_{1+s}(\widehat{u}),$$

and $\exists g \in C^{1,\gamma}(\mathbb{R}^{n-1})$, such that

$$B_\eta(x_0) \cap \widehat{\Gamma}(\widehat{u}) = B_\eta(x_0) \cap \{x_n \leq g(x')\},$$

after a possible rotation in $\mathbb{R}^n$.

Reduction to an obstable problem for the fractional Laplacian without drift

We assume $s \in (1/2, 1)$, $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$, $0 \leq c \in C^s(\mathbb{R}^n)$, $\widehat{\varphi} \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

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Given $\widehat{u} \in C^{1,s}(\mathbb{R}^n)$ a solution of (0.1), let w solve

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$$(-\Delta)^s w = \langle b(x), \nabla \widehat{u}(x) \rangle + c(x)\widehat{u}(x).$$

Then $w \in C^{3s}(\mathbb{R}^n)$. If $u := \widehat{u} - w$ and $\varphi := \widehat{\varphi} - w$, then u solves the obstacle problem *without drift*

$$\min\{(-\Delta)^s u(x), u(x) - \varphi(x)\} = 0, \forall x \in \mathbb{R}^n. \tag{0.2}$$

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Remark: φ can only be assumed to be in $C^{3s}(\mathbb{R}^n)$, even if $\widehat{\varphi}$ is smooth. Notice that $\Gamma(u) := \partial\{u = \varphi\} = \widehat{\Gamma}(\widehat{u})$. We also define $\Gamma_{1+s}(u) = \widehat{\Gamma}_{1+s}(\widehat{u})$.

Reduction of our main result: $C^{1,\gamma}$ regularity of $\Gamma_{1+s}(u)$

Theorem (Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016)

Let $s \in (1/2, 1)$ and $\varphi \in C^{3s}(\mathbb{R}^n)$. Let u solve (0.2) and $x_0 \in \Gamma_{1+s}(u)$. Then $\exists \gamma \in (0,1)$ and $\eta > 0$, such that

$$B_\eta(x_0) \cap \Gamma(u) \subseteq \Gamma_{1+s}(u),$$

and $\exists g \in C^{1,\gamma}(\mathbb{R}^{n-1})$, such that

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after a possible rotation in $\mathbb{R}^n$.

Since $\widehat{\Gamma}_{1+s}(\widehat{u}) = \Gamma_{1+s}(u)$, this implies the $C^{1,\gamma}$ regularity of $\widehat{\Gamma}_{1+s}(\widehat{u})$, our main result.

Literature

Caffarelli, Salsa, Silvestre (2008): assuming $\varphi \in C^{2,1}(\mathbb{R}^n)$:
$C^{1,\gamma}$ regularity of $\Gamma_{1+s}(u)$ (case of no drift).

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Garofalo, Petrosyan, Pop & SVG: assuming $\varphi \in C^{3s}(\mathbb{R}^n)$:

$C^{1,\gamma}$ regularity of $\Gamma_{1+s}(u)$ (case of no drift, which implies that $\widehat{\Gamma}_{1+s}(\widehat{u})$ is also $C^{1,\gamma}$).

Localize our problem

Let $a := 1 - 2s$. Given $v \in C^2(\mathbb{R}^n \times \mathbb{R}_+)$, define the operator L_a as

$$L_a v(x, y) = \text{div}(|y|^a \nabla v)(x, y), \ \forall \ (x, y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

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Why are we looking at this operator? If $L_a w = 0$, then

$$\lim_{y \downarrow 0} |y|^a w_y(x,y) = -(-\Delta)^s w(x,0),$$

i.e., $(-\Delta)^s$ is a Dirichlet-to-Neumann map for the operator L_a.

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i.e., $(-\Delta)^s$ is a Dirichlet-to-Neumann map for the operator L_a.
Given $x_0 \in \Gamma(u)$, define

$$v_{x_0}(x, y) := u(x, y) - \varphi(x, y) - \frac{1}{2s}(-\Delta)^s \varphi(x_0)|y|^{1-a},$$

where $u(x, y), \varphi(x, y)$ are the L_a-harmonic extensions of $u(x)$ and $\varphi(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}_+$.

Localize our problem

Then

$$\begin{aligned} &L_a v_{x_0} = 0 \text{ on } \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}), \\ &v_{x_0} \geq 0 \text{ on } \mathbb{R}^n \times \{0\}, \\ &L_a v_{x_0}(x,y) \leq h_{x_0}(x)\mathcal{H}^n|_{\{y=0\}} \text{ on } \mathbb{R}^{n+1}, \\ &L_a v_{x_0}(x,y) = h_{x_0}(x)\mathcal{H}^n|_{\{y=0\}} \text{ on } \mathbb{R}^{n+1} \setminus (\{y=0\} \cap \{v_{x_0} = 0\}). \end{aligned} \tag{0.3}$$

where $h_{x_0}(x) := 2((-\Delta)^s\varphi(x) - (-\Delta)^s\varphi(x_0))$.

Optimal regularity

Petrosyan & Pop (2016)

By means of a new monotonicity formula, established the optimal $C^{1,s}(\mathbb{R}^n)$ regularity when $s \in (1/2, 1)$, $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$, $0 \leq c \in C^s(\mathbb{R}^n)$, $\widehat{\varphi} \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

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Historical background: Almgren's monotonicity formula

The crucial tool introduced to establish the regularity of $\widehat{u}$ is a fundamental monotonicity formula proved in 1979 by F. Almgren, who showed that if $\Delta u = 0$ in B_1, then the frequency of u, given by

$$r \to N(u, r) = \frac{rD(r)}{H(r)} = \frac{r \int_{B_r} |\nabla u|^2}{\int_{S_r} u^2},$$

is increasing in $(0, 1)$. Furthermore, $N(r) \equiv \kappa \iff u$ is homogeneous of degree κ, i.e., $u(rx) = r^\kappa u(x)$.

Crucial tool in the proof of the optimal regularity

Recall that

$$v_{x_0}(x,y) := u(x,y) - \varphi(x,y) - \frac{1}{2s}(-\Delta)^s\varphi(x_0)|y|^{1-a}$$

Crucial tool in the proof of the optimal regularity

Recall that

$$v_{x_0}(x,y) := u(x,y) - \varphi(x,y) - \frac{1}{2s}(-\Delta)^s\varphi(x_0)|y|^{1-a}$$

Theorem (Monotonicity of the frequency)

Let $s \in (1/2, 1)$, $\alpha \in (1/2, s)$ and $x_0 \in \Gamma(u)$. Then $\forall p \in [s, \alpha + s - 1/2)$, $\exists C, \gamma, r_0 > 0$ such that $(0, r_0) \ni r \mapsto e^{Cr^\gamma}\Phi^p_{x_0}(r) \nearrow$, where

$$\Phi^p_{x_0}(r) := r\frac{d}{dr}\log\max\left\{\int_{S_r}|v_{x_0}(x_0+\cdot)|^2|y|^{1-2s}, r^{n+3+2(p-s)}\right\}$$

Moreover,

$$\Phi^p_{x_0}(0+) \in \Big\{n+3, n+3+2(p-s)\Big\} \cup [n+5-2s, \infty).$$

The regular set

Definition (Regular point for u)

We say that $x_0 \in \Gamma(u)$ is *regular* if

$$\Phi^p_{x_0}(0+) = n + 3, \ \forall p \in (s, 2s - 1/2).$$

We write $\Gamma_{1+s}(u) := \{x_0 \in \Gamma(u) \mid \Phi^p_{x_0}(0+) = n + 3, \ \forall p \in (s, 2s - 1/2)\}$.

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Definition (Regular point for $\widehat{u}$)

We say that $x_0 \in \widehat{\Gamma}(\widehat{u})$ is *regular* if $x_0 \in \Gamma_{1+s}(u)$ and we denote the set of regular points of $\widehat{u}$ as $\widehat{\Gamma}_{1+s}(\widehat{u})$. (So $\Gamma_{1+s}(u) = \widehat{\Gamma}_{1+s}(\widehat{u})$)

Regularity of the regular part of the free boundary

Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016:

$\Gamma_{1+s}(u)$ is locally a $C^{1,\gamma}$-regular surface. As a consequence, the same result holds for $\widehat{\Gamma}_{1+s}(\widehat{u})$.

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Our central results are:

- a new Weiss type monotonicity formula and
- a new epiperimetric inequality,

both inspired by those originally obtained by Weiss for the classical obstacle problem.

Spaces

- Weighted Hölder spaces: let $a \in (0,1)$, $\Omega \subset \mathbb{R}^n \times \mathbb{R}_+$ an open set. $u \in C^1(\Omega)$ is in $C_a^{1,\alpha}(\overline{\Omega})$ if

$$||u||_{C^\alpha(\overline{\Omega})} + ||u_{x_i}||_{C^\alpha(\overline{\Omega})} + |||y|^a \partial_y u||_{C^\alpha(\overline{\Omega})} < \infty.$$

- Weighted Sobolev space: let $U \subset \mathbb{R}^{n+1}$ be a Borel measurable set. $w \in H^1(U, |y|^a)$ if $w, Dw \in L^2_{\text{loc}}(U)$ and

$$\int_U (|w|^2 + |\nabla w|^2)|y|^a < \infty.$$

Outline of our approach

1. First main ingredient consists of the "almost monotonicity" of a Weiss-type functional which intuitively measures the closeness of the solution v to the prototypical homogeneous solution of degree $(1+s)$, i.e., the function

$$\left(x_n + \sqrt{x_n^2 + y^2}\right)^s \left(x_n - s\sqrt{x_n^2 + y^2}\right)$$

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2. The second central ingredient is an epiperimetric inequality for the Weiss functional.

The combination of these results provides us with a powerful tool to establish a geometric rate of decay for the Weiss functional, which in turn allows us to study the homogeneous blowups of v, $v_r(x,y) = \frac{v(rx,ry)}{r^{1+s}}$, and prove the $C^{1,\gamma}$ regularity of $\Gamma_{1+s}(u)$.

In more detail

Given $x_0 \in \Gamma_{1+s}(u)$, our goals will be to prove that:

- The homogeneous rescalings

$$v_{x_0,r}(x,y) = \frac{v_{x_0}(x_0 + rx, ry)}{r^{1+s}}$$

converge to a (unique) solution of (0.3) with h_{x_0} substituted by 0, which is homogeneous of degree $(1+s)$.

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- We can express the unique homogeneous blowup at x_0, $v_{x_0,0}(x,y)$, as

$$a_{x_0}\left(\langle x, e_{x_0}\rangle + \sqrt{\langle x, e_{x_0}\rangle^2 + y^2}\right)^s \left(\langle x, e_{x_0}\rangle - s\sqrt{\langle x, e_{x_0}\rangle^2 + y^2}\right),$$

for some $e_{x_0} \in S_1'$. Moreover, $v_{x_0,0}$ is nonzero.

In more detail

- To prove $C^{1,\gamma}$ regularity of $\Gamma_{1+s}(u)$:
  $|a_{\bar{x}} - a_{\bar{y}}| \leq C|\bar{x} - \bar{y}|^{\gamma}, \quad |e_{\bar{x}} - e_{\bar{y}}| \leq C|\bar{x} - \bar{y}|^{\gamma}$.

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To obtain the last inequalities we first prove that $\exists \eta_0 = \eta_0(x_0) > 0$ such that

$$\int_{S_1} \left| v_{\bar{x},0} - v_{\bar{y},0} \right| |y|^a \leq C|\bar{x} - \bar{y}|^{\gamma} \text{ for } \bar{x}, \bar{y} \in B'_{\eta_0}(x_0) \cap \Gamma(u),$$

where C and $\gamma > 0$ are universal constants.

First main ingredient: Weiss type monotonicity formula

Weiss type functional:

$$W_L(v, r, x_0) = W_L(r)$$
$$= \frac{1}{r^{n+2}} \left[\int_{B_r} |\nabla v_{x_0}|^2 |y|^a + \int_{B_r'} v_{x_0} h_{x_0} \right] - \frac{1+s}{r^{n+3}} \int_{S_r} |v_{x_0}(x_0 + \cdot)|^2 |y|^a.$$

First main ingredient: Weiss type monotonicity formula

Theorem (Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016)

Assume $x_0 \in \Gamma_{1+s}(u)$. There exists a universal constant $C > 0$ such that

$$\frac{d}{dr}\left(W_L(v,r) + Cr^{2s-1}\right) \geq \frac{2}{r^{n+2}} \int_{S_r} \left(\langle \nabla v_{x_0}, \nu \rangle - \frac{(1+s)v_{x_0}}{r} \right)^2 |y|^a. \tag{0.4}$$

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Hence, $r \mapsto W_L(v,r) + Cr^{2s-1}$ is $\nearrow$, so the following limit exists:

$$W_L(v, 0+) \stackrel{def}{=} \lim_{r \to 0} W_L(v,r).$$

Blow-ups

We recall the definition of the homogeneous rescallings of v:

$$v_r(x,y) = \frac{v(rx,ry)}{r^{1+s}}.$$

Lemma (Convergence to blow-ups)

Let $0 \in \Gamma_{1+s}(u)$. Given $r_j \to 0, \exists v_0 \in C^{1,\alpha}_{\mathrm{a,loc}}((\mathbb{R}^n)^{\pm} \cup \{y=0\}), \forall \alpha \in (0,1)$, such that

$$v_{r_j} \to v_0 \text{ in } C^{1,\alpha}_{\mathrm{a,loc}}((\mathbb{R}^n)^{\pm} \cup \{y=0\}).$$

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We still do not have: uniqueness of the blow up, $v_0 \neq 0$ and a rate of convergence of the recallings to the blow-ups.

Second main ingredient: Epiperimetric inequality

Let

$$\widehat{v_0}(x,y) := \left(x_n + \sqrt{x_n^2+y^2}\right)^s \left(x_n - s\sqrt{x_n^2+y^2}\right).$$

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When $h_{x_0} \equiv 0$ and $r = 1$, our Weiss functional takes an easier form:

$$W_L(v) := W_L(v,1) = \int_{B_1} |\nabla v|^2 |y|^a - (1+s) \int_{S_1} v^2 |y|^a d\mathcal{H}^{n-1}.$$

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Theorem (Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016)

There exists $\kappa, \theta \in (0,1)$ such that if $w \in H^1(B_1, |y|^a)$ is homogeneous of degree $(1+s)$, $w \geq 0$ on B_1' and $\|w - \widehat{v_0}\|_{H^1(B_1,|y|^a)} \leq \theta$, then there exists $\zeta \in H^1(B_1, |y|^a)$ such that $\zeta = w$ on S_1, $\zeta \geq 0$ on B_1' and

$$W_L(\zeta) \leq (1-\kappa) W_L(w).$$

To recall is to live :)

Recall that our goal is to prove that $\exists \eta_0 = \eta_0(x_0) > 0$ such that

$$\int_{S_1} \left| v_{\bar{x},0} - v_{\bar{y},0} \right| |y|^a \leq C|\bar{x} - \bar{y}|^\gamma \text{ for } \bar{x}, \bar{y} \in B'_{\eta_0}(x_0) \cap \Gamma(u),$$

where C and $\gamma > 0$ are universal constants.

Uniqueness of blowups (for points close to $x_0 \in \Gamma_{1+s}(u)$)

Let $r > 0$, $x_0 \in \Gamma_{1+s}(u)$, $v_{x_0,r}(x,y) = \frac{v_{x_0}(x_0+rx,ry)}{r^{1+s}}$.

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There exist constants $r_0 = r_0(x_0), \eta_0 = \eta_0(x_0) > 0$ such that

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Moreover, if $v_{\bar{x},0}$ is any blow up of v at $\bar{x} \in \Gamma(u) \cap B'_{\eta_0}(x_0)$, then

$$\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},0}||y|^a \leq Cr^\gamma, \text{ for all } r \in (0, r_0),$$

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where $C > 0$ and $\gamma \in (0,1)$ are universal constants. In particular, the blow-up limit $v_{\bar{x},0}$ is unique.

The main ingredient in the proof is the fact that $W_L(v,r) \leq Cr^{\gamma}$, which is proved by using the epiperimetric inequality.

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- The unique blowup $v_{\bar{x},0}$ above is nonzero.

Analysing blowups for $\bar{x}, \bar{y} \in \Gamma(u)$ close to $x_0 \in \Gamma_{1+s}(u)$

Proposition

Assume $x_0 \in \Gamma_{1+s}(u)$. Then, there exists $\eta_0 = \eta_0(x_0) > 0$ such that

$$\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}||y|^a \leq C|\bar{x} - \bar{y}|^\gamma \quad \text{for } \bar{x}, \bar{y} \in B'_{\eta_0}(x_0) \cap \Gamma(u),$$

where C and $\gamma > 0$ are universal constants.

$C^{1,\gamma}$ regularity of the regular free boundary

Theorem (Garofalo, Petrosyan, Pop & Smit Vega Garcia, 2016)

Let $s \in (1/2, 1)$, $b \in C^s(\mathbb{R}^n; \mathbb{R}^n)$, $0 \leq c \in C^s(\mathbb{R}^n)$ and $\widehat{\varphi} \in C^{3s}(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. Let $\widehat{u} \in C^{1,s}(\mathbb{R}^n)$ solve (0.1) and $x_0 \in \widehat{\Gamma}_{1+s}(\widehat{u})$. Then $\exists \gamma \in (0,1)$ and $\eta > 0$, such that

$$B'_\eta(x_0) \cap \widehat{\Gamma}(\widehat{u}) \subseteq \widehat{\Gamma}_{1+s}(\widehat{u}),$$

and $\exists g \in C^{1,\gamma}(\mathbb{R}^{n-1})$, such that

$$B'_\eta(x_0) \cap \widehat{\Gamma}(\widehat{u}) = B'_\eta(x_0) \cap \{x_n \leq g(x')\},$$

after a possible rotation in $\mathbb{R}^n$.

Thank you!