# Regularity of solutions to divergence form complex p-elliptic operators

Jill Pipher
Brown University

MSRI: January, 2017

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$$\triangle u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

Note: $\triangle u = \operatorname{div} A\nabla$ where $A = Id$ and we will be considering variable coefficient operators, where the matrix $A$ shares certain structural properties such as positivity or ellipticity.

# Non-smooth Dirichlet data

The solution to Laplace's equation in $\mathbb{R}^n_+$ with data $g(x)$ on the boundary $t = 0$ is given by an explicit formula: the Poisson extension $u(x,t) = \int_{\mathbb{R}^{n-1}} g(y)P_t(x-y)dy$. And this integral may converge even when $g$ is only (Lebesgue) measurable $g$, not continuous: for example, if $g \in L^\infty$, then $u$ is bounded and harmonic. And $u$ converges weak-star to $g$.

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In fact, the Poisson integral of an $L^p$ function for $1 \leq p \leq \infty$ makes sense, and satisfies the estimate:

$$\lim_{t \to 0} \int_{\mathbb{R}} |u(x,t)|^p dx = \int_{\mathbb{R}} |g(x)|^p dx$$

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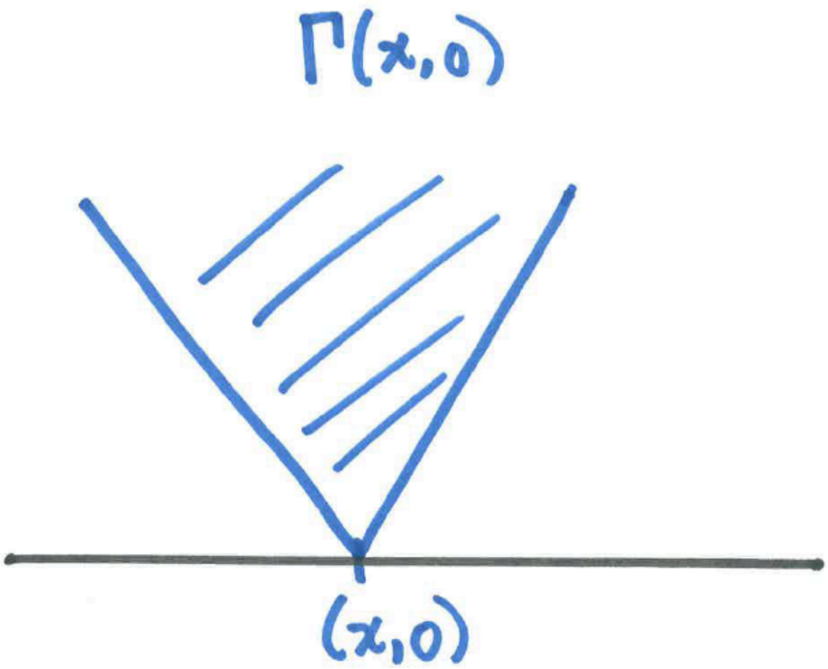
When $1 < p$ we have: $u$ is the Poisson integral of an $L^p$ function *if and only if*

$N(u)(x) \in L^p$ and $u$ converges in $L^p$ and pointwise almost everywhere *nontangentially* to its boundary data.

# Nontangential convergence

Let $\Gamma(x,0) = \{(x', y) : |x - x'| < cy\}$ denote the cone at $(x, 0)$:

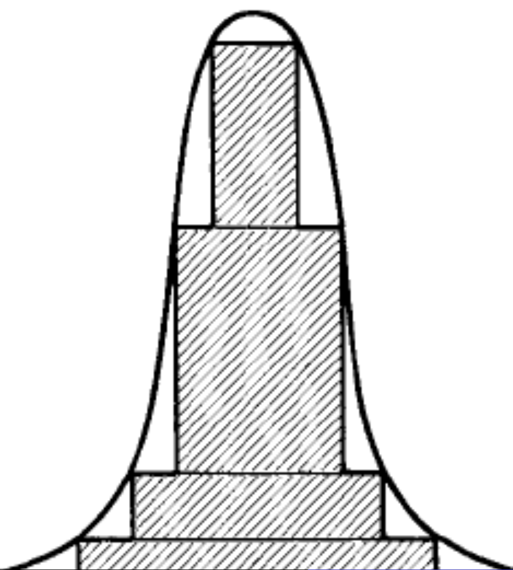


# Dirichlet problem on $\mathbb{R}^{\ltimes}_{+}$

$$u(x,t) = \int_{\mathbb{R}} g(y) P_t(x-y) dy,$$

and

$$Nu(x) = \sup\{(u(x',t) : (x',t) \in \Gamma(x,0)\}$$



# Nontangential maximal functions and the Dirichlet problem

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The Dirichlet problem with data in $L^p$ is uniquely solvable for Laplace's equation when $p > 1$ in smooth domains:

$$\Delta u = 0 \in \mathbb{R}^{n+1}_+, \;\; u(x, 0) = f(x) \in L^p(\mathbb{R}^n)$$

with

$$\|Nu\|_p \leq C\|f\|_p$$

This apriori estimate for continuous $f \in L^p(\mathbb{R}^n)$ implies that solutions to the $L^p$ Dirichlet problem converge nontangentially to their boundary values.

# Boundary value problems for second order divergence form operators

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$$\lambda|\xi|^2 \leq \langle A(X)\xi,\xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \quad (1)$$

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*Since the coefficients of $A$ are not differentiable, what does $Lu = 0$ mean?*

$$\int_{\mathbb{R}^n_+} A(X)\nabla u.\nabla\phi dX = 0$$

*for all appropriate test functions $\phi$ and for all $u$ with square integrable derivatives.*

# Motivation for studying regularity of solutions, and sharp boundary value problems

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- Geometry of boundary of a domain properties of the harmonic/elliptic measure (Kenig-Toro, Milakis-Pipher-Toro,

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- For both systems and higher order operators, theory is not as well developed. (Lack of: Positivity, maximum principles, and the existence of a boundary measure)
- When matrix $A$ has complex coefficients: some milestones, and some partial progress: Kato square root problem is a Regularity/Neumann boundary value problem (Auscher - Hofmann - Lacey - McIntosh - Tchamitchian); perturbations of operators .

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- The solvability of the Dirichlet problem with data in $L^p$ is equivalent to a real-variable property of the measure, which in turn depends on the smoothness of the coefficients and the geometry of the domain.
- None of this applies in the bounded, measurable complex-coefficient setting.

# Complex valued elliptic operators

Program: The study of solutions to operators of the form

$$L := -\operatorname{div} A(x,t)\nabla, \quad (x,t) \in \mathbb{R}^n_+$$

where $A$ may be complex valued, and the natural boundary value problems associated with them.

Some results in the complex setting takes place under the assumption that solutions to $L$ satisfy DeG-N-M bounds. Other work focuses on structural assumptions on these operators. In [Dindos,P. 2016] we take the latter approach to develop a theory of regularity of solutions to complex coefficient operators and use this to solve certain boundary value problems.

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Let $A$ by an $n \times n$ matrix of complex valued coefficients satisfying $\lambda|\zeta|^2 < Re(A\zeta, \zeta)$ and $(A\zeta, \zeta) < \Lambda|\zeta|^2$, for some real and positive $\lambda$ and $\Lambda$.

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The Kato square root problem (as re-formulated by McIntosh) asks about the domain of $\sqrt{L}$, namely whether one has the estimate $\|\sqrt{L}(f)\|_{L^2} \lesssim \|\nabla_x f)\|_{L^2}$.

# Structural assumptions

The estimate on $\sqrt{L}$ is equivalent to solving an $L^2$ Regularity problem for the operator, $\tilde{L}$ below, or a Neumann problem for $\tilde{L}^*$, where the matrix for $\tilde{L}$ in dimension $n+1$ is

$$\tilde{A} = \left[\begin{array}{c|c} A & \vec{0} \\ \hline \vec{0} & 1 \end{array}\right]$$

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For $\tilde{L}$ as above in block form, the family of operators $\{e^{-t\sqrt{L}}\}$ is the Poisson semigroup: solutions to $\tilde{L}u = 0$ in $\mathbb{R}^{n+1}_+$ with data $f(x) \in \mathbb{R}^n$ are given by $\{e^{-t\sqrt{L}}f(x)\}$, and are uniformly bounded in $L^2$ for all $t$ by the $L^2$ of the norm of the data. (The Dirichlet problem is solvable in a larger range of $p$ [Mayboroda, 2010].

# Structural assumptions and p-ellipticity

In a series of papers, Cialdea and Maz'ya define a notion they term $L^p$-dissipativity, motivated by understanding when semigroups generated by second order elliptic operators are contractive in $L^p$. (Always true for real second order elliptic operators.)

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Our condition, termed *p-ellipticity* in a very recent paper [Carbonaro, Dragičević], is a slight strengthening of $L^p$-dissipativity.

The matrix $A$ is *p-elliptic* if

$$|1 - 2/p| < \mu(A)$$

where

$$\mu(A) = \operatorname*{ess\,inf}_{(x,\xi)\in\Omega\times\mathbb{C}^n\setminus\{0\}} Re\, \frac{\langle A(x),\xi,\xi\rangle}{|\langle A(x),\xi,\bar{\xi}\rangle|}.$$

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i\frac{\beta}{p'}$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$. [CD] shows that the matrix $A$ is $p$-elliptic iff for a.e. $x \in \Omega$

$$\Re e \langle A(x)\xi, \mathcal{J}_p\xi \rangle \geq \lambda_p |\xi|^2, \qquad \forall \xi \in \mathbb{C}^n \tag{2}$$

for some $\lambda_p > 0$.

**Theorem**

*Assume that the matrix $A$ is p-elliptic. Then there exists $\lambda'_p = \lambda'_p(\lambda, \Lambda, \lambda_p) > 0$ such that for any nonnegative, bounded and measurable function $\chi$ and any $u$ such that $|u|^{(p-2)/2}u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$, we have*

$$\mathcal{R}e \int_\Omega \langle A(x)\nabla u, \nabla(|u|^{p-2}u)\rangle \chi(x)\, dx \geq \lambda'_p \int_\Omega |u|^{p-2}|\nabla u|^2 \chi(x)\, dx. \tag{3}$$

We also observe:
For all $p > 1$, and for all $x$ for which $u(x) \neq 0$

$$|\nabla(|u(x)|^{p/2-1}u(x))|^2 \approx |u(x)|^{p-2}|\nabla u(x)|^2.$$

# Regularity result

Suppose that $u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$ is the weak solution to the operator $\mathcal{L}u := \mathrm{div} A(x)\nabla u + B(x) \cdot \nabla u = 0$ in $\Omega$. Let $p_0 = \inf\{p > 1 : A \text{ is } p\text{-elliptic}\}$, and suppose that $B$ has measurable coefficients $B_i \in L^{\infty}_{loc}(\Omega)$ satisfying the condition

$$|B_i(x)| \leq K(\delta(x))^{-1}, \quad \forall x \in \Omega \tag{4}$$

where the constant $K$ is uniform, and $\delta(x)$ denotes the distance of $x$ to the boundary of $\Omega$.

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Then we have the following improvement in the regularity of $u$. For any $B_{4r}(x) \subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\int_{B_r(x)} |u|^p \, dy\right)^{1/p} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u|^q \, dy\right)^{1/q} + \varepsilon \left(\int_{B_{2r}(x)} |u|^2 \, dy\right)^{1/2} \tag{5}$$

for all $p, q \in (p_0, \frac{p_0' n}{n-2})$. (Here $p_0' = p_0/(p_0 - 1)$ and when $n = 2$ one can take $p, q \in (p_0, \infty)$.)

# Regularity, continued

The constant in the estimate depends on the dimension, the $p$-ellipticity constants, $\Lambda$, $K$ and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or $u$.

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The constant in the estimate depends on the dimension, the $p$-ellipticity constants, $\Lambda$, $K$ and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or $u$. Moreover, for all $p \in (p_0, p_0')$ and any $\varepsilon > 0$

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \iint_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left( \int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2}$$

where the constants depend only on the dimension, $p$, $\Lambda$, $K$ and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega; \mathbb{C})$.

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where the constants depend only on the dimension, $p$, $\Lambda$, $K$ and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega; \mathbb{C})$. The range in the reverse Hölder is sharp: Mayboroda gives a counterexample when $q = 2$ for any $p > \frac{2n}{n-2}$ under the assumption of 2-ellipticity.

[Dindos-P., 2016] Let $1 < p < \infty$, and let $\Omega$ be the upper half-space $\mathbb{R}^n_+ = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. Consider the operator

$$\mathcal{L}u = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$$

and assume that the matrix $A$ is $p$-elliptic with constants $\lambda_p, \Lambda$ and $\mathcal{I}m\, A_{0j} = 0$ for all $1 \leq j \leq n-1$ and $A_{00} = 1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} \left[|\nabla A(x)|^2 + |B(x)|^2\right] \delta(x)\, dx \tag{6}$$

is a Carleson measure in $\Omega$. Let us also denote

$$d\mu'(x) = \sup_{B_{\delta(x)/2}(x)} \left[\sum_j |\partial_0 A_{0j}|^2 + \left|\sum_j \partial_j A_{0j}\right|^2 + |B(x)|^2\right] \delta(x)\, dx. \tag{7}$$

Then there exist $K = K(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ and $C(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ such that if

$$\|\mu'\|_{\mathcal{C}} < K \tag{8}$$

then the $L^p$ Dirichlet problem is solvable for $\mathcal{L}$

By solvability of the $L^p$-Dirichlet problem, we mean

$$\|\tilde{N}_{p,a}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega;\mathbb{C})}$$

where

$$\tilde{N}_{p,a}(u)(Q) := \sup_{x\in\Gamma_a(Q)} w(x)$$

with

$$w(x) := \left(\int_{B_{\delta(x)/2}(x)} |u(z)|^p \, dz\right)^{1/p}.$$

## Corollary

*Suppose the operator* $\mathcal{L}$ *on* $\mathbb{R}^n_+$ *has the form*

$$\mathcal{L}u = \partial_0^2 u + \sum_{i,j=1}^{n-1} \partial_i (A_{ij} \partial_j u)$$

*where the matrix* $A$ *has coefficients satisfying the Carleson condition.*
*Then for all* $1 < p < \infty$ *for which* $A$ *is p-elliptic, the* $L^p$*-Dirichlet problem is solvable for* $\mathcal{L}$ *and the estimate*

$$\|\tilde{N}_{p,a} u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega;\mathbb{C})} \tag{9}$$

*holds for all energy solutions* $u$ *with datum* $f$*.*

## Definition

For $\Omega \subset \mathbb{R}^n$ as above, the square function of some $u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_a(u)(Q) := \left( \int_{\Gamma_a(Q)} |\nabla u(x)|^2 \delta(x)^{2-n}\, dx \right)^{1/2} \tag{10}$$

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## Definition

[Dindos-Petermichl-P.] For $\Omega \subset \mathbb{R}^n$, the $p$-adapted square function of $u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_{p,a}(u)(Q) := \left( \int_{\Gamma_a(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n}\, dx \right)^{1/2} \qquad (11)$$

# Regularity when $p > 2$

## Lemma

*Let the matrix A be p-elliptic for $p \geq 2$ and let B have coefficients satisfying $|B_i(x)| \leq K(\delta(x))^{-1}, \quad \forall x \in \Omega$ Suppose that u is a $W^{1,2}_{loc}(\Omega;\mathbb{C})$ solution to $\mathcal{L}$ in $\Omega$. Then, for any ball $B_r(x)$ with $r < \delta(x)/4$,*

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^p dy \tag{12}$$

*and*

$$\left( \iint_{B_r(x))} |u(y)|^q dy \right)^{1/q} \lesssim \left( \iint_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \tag{13}$$

*for all $q \in (2, \frac{np}{n-2}]$ when $n > 2$, and where the implied constants depend only p-ellipticity and K. When n = 2, q can be any*

# Sketch of proof

Let $v = u\varphi$ where $\varphi$ is a cut-off function function associated to the ball $B_r(x)$, and compute

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla\varphi + A^*\nabla u \cdot \nabla\varphi.$$

# Sketch of proof

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Multiply both sides of this equation by $|v|^{p-2}\overline{v}$ and integrate by parts to obtain:

$$
\begin{aligned}
\int \nabla(|v|^{p-2}\overline{v}) \cdot A\nabla v \, dy = & \int (|v|^{p-2}\overline{v}) B \cdot \nabla v \, dy \\
& + \int \nabla(|v|^{p-2}\overline{v} u) \cdot A\nabla \varphi \, dy \\
& - \int |v|^{p-2}\overline{v} u \, B \cdot \nabla \varphi dy \\
& - \int |v|^{p-2}\overline{v} A\nabla u \cdot \nabla \varphi \, dy \\
& - \int |v|^{p-2}\overline{v} A^* \nabla u \cdot \nabla \varphi \, dy
\end{aligned}
$$

By $p$-ellipticity, the real part of the left hand side is bounded from below by $\lambda_p \int |v|^{p-2}|\nabla v|^2 \, dy$.

Each term is treated separately. For example, the first of the five terms on the right hand side above has the bound

$$\left|\int (|v|^{p-2}\overline{v}) \cdot B\nabla v \, dy\right| \lesssim Kr^{-1} \left(\int |v|^{p-2}|\nabla v|^2 \, dy\right)^{1/2} \left(\int |v|^p \, dy\right)^{1/2}$$

which yields

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^p dy$$

The Sobolev embedding gives

$$\left(\int_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{2r}(x)} |v|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(r^2 \int_{B_{2r}(x)} |\nabla(|v|^{p/2-1} v)|^2\, dy\right)^{1/p}$$

where $\tilde{p} = \frac{pn}{n-2}$.

This gives a reverse Hölder inequality for $u$. That is,

$$\left(\int_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{\alpha r}(x)} |u|^{p}\, dy\right)^{1/p}$$

which can be iterated $k$ times to give

$$\left(\int_{B_r(x)} |u|^{p_k}\, dy\right)^{1/p_k} \lesssim \left(\int_{B_{\alpha^k r}(x)} |u|^{2}\, dy\right)^{1/2}$$

for $p_k = 2(\frac{n}{n-2})^k$, as long as $p_{k-1} < p$.

# The $L^p$ Dirichlet problem

From now on, in addition to $p$-ellipticity, assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2]\delta(x)\,dx$$

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# The $L^p$Dirichlet problem

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is a Carleson measure in $\Omega$. Sometimes, and for certain coefficients of $A$, we will assume that their Carleson norm $\|\mu\|_{\mathcal{C}}$ is small. The Carleson measure conditions on the coefficients of $\mathcal{L}$, as well as $p$-ellipticity of $A$, are compatible with a useful change of variables that is a bijection from $\overline{\mathbb{R}^n_+}$ onto $\overline{\Omega}$.

# Assumptions on the coefficients, explained

Some observations on the structural assumptions made for solvability of the Dirichlet problem. It suffices to formulate the result in the case $\Omega = \mathbb{R}^n_+$ by using the pull-back map alluded to above. Because the coefficients are required to have *small* Carleson norm this puts a restriction on the size of the Lipschitz constant of the map that defines the domain $\Omega$.

For technical reasons we also required that all coefficients $A_{0j}$, $j = 0, 1, \ldots, n-1$ are real. This can be ensured as follows. When $j > 0$:

$$\partial_0([\mathcal{I}m\, A_{0j}]\partial_i u) = \partial_j([\mathcal{I}m\, A_{0j}]\partial_0 u)+(\partial_0[\mathcal{I}m\, A_{0j}])\partial_i u-([\partial_i \mathcal{I}m\, A_{0j}])\partial_0 u$$

which allows one to move the imaginary part of the coefficient $A_{0j}$ onto the coefficient $A_{j0}$ at the expense of two first order terms. However, this does not work for the coefficient $A_{00}$.

We will require that $A_{00}$ is real, then a multiplication of the coefficients of $\mathcal{L} = \partial_i \left( A_{ij}(x)\partial_j \right) + B_i(x)\partial_i$ by $\alpha = A_{00}^{-1}$ reduces one to $A_{00} = 1$. When $\alpha$ is real (or when $\mathcal{I}m\, \alpha$ is sufficiently small) $p$-ellipticity of $A$ is equivalent to $p$-ellipticity of the new operator.

if $\mathcal{I}m\, \alpha$ is not small, the $p$-ellipticity, after multiplication of $A$ by $\alpha$ may not be preserved. Thus, in the most case, one must assume the $p$-ellipticity of the new matrix $\tilde{A}$ which has all coefficients $\tilde{A}_{0j}$, $j = 0, 1, \ldots, n-1$ real.

The proof proceeds by establishing, through an integration by parts and stopping time argument, the equivalence of the $p$-adapted square function and the $p$-averaged nontangential maximal function. The connection to $p$-ellipticity is made in the following estimate:

$$\lambda_p' \iint_{\mathbb{R}^n_+} |\nabla u|^2 |u|^{p-2} x_0 \, dx' \, dx_0 \le \int_{\mathbb{R}^{n-1}} |u(0,x')|^p \, dx' + C\|\mu'\|_{\mathcal{C}} \int_{\mathbb{R}^{n-1}} \left[\tilde{N}_{p,a}(u)\right]^p \, dx'.$$

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