Harmonic Analysis Techniques in Several Complex Variables

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Credits

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Objects

- The Cauchy Integral along the boundary of a (simply connected) planar domain $D \subset \mathbb{C}$:

$$\mathcal{C}f(z) \;=\; \frac{1}{2\pi i}\int\limits_{w\in bD} \frac{f(w)}{w-z}\,dw\,, \qquad z \in D$$

More precisely, we regard $\mathcal{C}$ as a Singular Integral Operator (SIO):

$$\mathcal{C}f(z) \;=\; p.v.\,\frac{1}{2\pi i}\int\limits_{w\in bD} \frac{f(w)}{w-z}\,dw\,, \qquad z \in bD$$

Landmark Results

- Theorem [Calderòn (1977); Coifman-McIntosh-Meyer (1982)]:

  Suppose $D \subset \mathbb{C}$ is a Lipschitz domain, i.e.

  $$bD = \{w = t + i\,A(t) \mid |A(t) - A(s)| \leq M|s - t|,\ s, t \in \mathbb{R}\}$$

  Then, the Cauchy Integral

  $$f \mapsto \mathcal{C}(f)$$

  is bounded: $L^p(bD, \sigma) \to L^p(bD, \sigma), \quad 1 < p < \infty$
  with respect to arc-length measure for bD
  (Here, $L^p(bD, \sigma) := \{f \mid \int_{bD} |f(w)|^p d\sigma(w) < \infty\},\ p > 1$)
- Theorem [Coifman-McIntosh-Meyer (1982)]:
  The Double Layer Potential Operator: $f \mapsto \mathcal{D}(f)$
  for a *Lipschitz domain $D \subset \mathbb{R}^N$ is bounded*:

  $$L^p(bD, \sigma) \to L^p(bD, \sigma), \quad 1 < p < \infty$$

Impact

- Elliptic Linear PDEs:
  Boundary Value Problems on non-smooth domains
- Harmonic Analysis: *New Techniques for SIOs*
- Geometric Function Theory: *Analytic Capacity*

- **One & Several complex variables**: *Orthogonal projections of L^2 onto spaces of holomorphic functions for domains:*

$$D \Subset \mathbb{C}^n, \quad n \geq 1$$

(Specifically, the Szegő projection and the Bergman projection, which map L^2 onto the holomorphic Hardy space (Szegő), and onto the Bergman space)

Motivation: L^p-regularity of orthogonal projections

E.g.,

- Holomorphic Hardy Space for $D \subset \mathbb{C}^n$, $n \geq 1$:

  $$H^p(bD,\sigma) := \left\{ F \,\middle|\, \overline{\partial} F(z) = 0, z \in D,\ \sup_{\epsilon>0} \int\limits_{z\in bD_\epsilon} |F(z)|^p d\sigma_\epsilon(z) < +\infty \right\}$$

  (A closed subspace of $L^p(bD,\sigma)$, $1 \leq p < \infty$).

- Pick $p = 2$: Orthogonal Projection
  $\mathbf{S} : L^2(bD,\sigma) \mapsto H^2(bD,\sigma)$:

  $$\mathbf{S} \text{ is orthogonal proj.} \iff \mathbf{S} = \mathbf{S}^* \iff \|\mathbf{S}\|_{L^2\to L^2} = 1$$

  ($\mathbf{S}$ = Szegő Projection)

- **L^p-Regularity problem for Szegő projection S:**

under minimal assumptions on D, find $P = P(D) \in [2, +\infty]$ so that

$\mathbf{S} : L^p(bD,\sigma) \to L^p(bD,\sigma)$ is bounded for all $P' < p < P$

L^p-regularity of Szegő projection: History and Motivation.

Size of (P', P) **is related to geometry and regularity of** D e.g.,

L. - Stein (2004):

- $n = 1$: If $D \Subset \mathbb{C}$ is Vanishing Chord-Arc (e.g., D of class C^1), then $P = +\infty$.
- $n = 1$: If $D \Subset \mathbb{C}$ is Lipschitz with constant M, then

$$P = 2\left(1 + \frac{\pi}{2 \arctan M}\right) > \ 4$$

- $n = 1$: If $D \Subset \mathbb{C}$ is a rectifiable local graph, then $P = 4$.

Connection with Cauchy Integral

- T (e.g., Cauchy int.) is also a projection: $L^2 \mapsto H^2$ i.e.,
    - T reproduces holomorphic functions from their boundary values (*"Cauchy formula"*)
    - T produces holomorphic functions from, say, C^1-smooth boundary data

- Compare T with the orthogonal projection $\mathbf{S}$:

$$\mathbf{S}T = T; \qquad T\mathbf{S} = \mathbf{S} \quad \Rightarrow \quad \mathbf{S}T^* = \mathbf{S}$$

$$\mathbf{S}(T^* - T) = \mathbf{S} - T$$

$$T = \mathbf{S}\,[I - (T^* - T)] \quad \text{on} \quad L^2 \qquad (I = \text{Identity op.})$$

The basic idea, after Kerzman & Stein

$$T = \mathbf{S}\,[I - (T^* - T)] \quad \text{on} \quad L^2 \tag{0.1}$$

- Basic idea: *if $T^* - T$ is "better" than T ("cancellation of singularities") then can use (0.1) to draw information: from* **S** *to T and vice-versa, from T to* **S**.
  - From **S** to T: another proof of $T : L^2 \to L^2$ (regularity of T).
  - From T to **S**: Suppose T bounded in L^2: can we solve (0.1) for **S**?

$$(T^* - T)^* = -(T^* - T)$$

$$\Longrightarrow$$

$$\mathbf{S} = T\,[I - (T^* - T)]^{-1} \quad \textbf{in} \quad L^2 \tag{0.2}$$

??? What about L^p, $p \neq 2$???

From T to **S** via: $\mathbf{S} = T\,[I - (T^* - T)]^{-1}$

Settings where we can deal with $p \neq 2$:

- $D \subset \mathbb{C}$ $(n = 1)$ and $T =$ Cauchy integral:
  - D of class C^2: $1 < p < \infty$ (Kerzman-Stein 1978), via: $T^* - T$ smoothing, which implies

    $[I - (T^* - T)]^{-1} : L^p \to L^p$
  - D vanishing-constant chord-arc: $1 < p < \infty$ (Semmes, 1983), via $T^* - T$ compact in L^p, which implies

    $[I - (T^* - T)]^{-1} : L^p \to L^p$

From T to $\mathbf{S}$ via: $T = \mathbf{S}\,[I - (T^* - T)]$

Settings where we can deal with $p \neq 2$:

- $D \subset \mathbb{C}^n$ $(n \geq 2)$ and T_ϵ = Henkin-Ramirez integral(s) (*later*):
  - D bounded, of class C^2 and strongly pseudo-convex (*later*): $1 < p < \infty$ (L. - Stein 2016), via
  - $T_\epsilon : \; L^p \to L^p \quad$ (*later*)
  - $T_\epsilon^* - T_\epsilon = A_\epsilon + B_\epsilon;$
  - $\|A_\epsilon\|_{L^p \to L^p} \leq C_p\, \epsilon\,; \qquad B_\epsilon : L^1 \to L^\infty$
  - $T_\epsilon = \mathbf{S}\,[I - A_\epsilon] - \mathbf{S}\,B_\epsilon$
  - Say $1 < p < 2$: $\quad \mathbf{S}B_\epsilon : \; L^p \hookrightarrow L^1 \to L^\infty \hookrightarrow L^2 \to L^2 \hookrightarrow L^p$
  - Choose $\epsilon = \epsilon(p)$ such that $\|A_\epsilon\|_{L^p \to L^p} < 1$:

$$\mathbf{S} = (T_\epsilon + \mathbf{S}B_\epsilon)\;[I - A_\epsilon]^{-1} : \; L^p \to L^p$$

Caveat: Studying the orthogonal projection $\mathbf{S}$ by comparing it with another operator T requires that the kernel of T be holomorphic as a function of the output parameter $z \in D$ ("holomorphic kernel") (which is of course the case when $n = 1$):

This talk is about holomorphic Cauchy-like kernels in complex dimension $n \geq 2$:

- Construction of holomorphic kernels
- L^p-regularity

Two crucial features of the 1-dimensional Cauchy Kernel

$$n = 1$$

- (as we just said) the fact that $H(w, z)$ is **holomorphic i.e., analytic,** as a function of $z \in D$ for fixed $w \in bD$;

- $H(w, z)$ is **universal**:

$$H(w, z) = \frac{1}{2\pi i}\frac{dw}{w - z}, \qquad z,\ w \in \mathbb{C} \times \mathbb{C} \setminus \{w = z\}$$

  in the sense that the effect of the particular domain $D \subset \mathbb{C}$ we are working with is only exerted through the inclusion $j : bD \hookrightarrow \mathbb{C}$, i.e.

$$H(w, z) = \frac{1}{2\pi i}\, j^*\!\left(\frac{dw}{w - z}\right)$$

A candidate for the Cauchy kernel in $\mathbb{C}^n$

$$n \geq 1$$

One option is to choose the Bochner-Martinelli kernel:

$$H(w,z) = \frac{1}{(2\pi i)^n} j^* \left(\sum_{\ell=1}^{n} \frac{\overline{w}_\ell - \overline{z}_\ell}{|w-z|^{2n}} \, dw_\ell \bigwedge_{\nu \neq \ell} d\overline{w}_\nu \wedge dw_\nu \right) \qquad (0.3)$$

- **Favorable features of BM-kernel**:
  - Bochner-Martinelli *is* a higher dim. analogue of Cauchy:

$$n = 1 \quad \Rightarrow \quad H(w,z) = \frac{1}{2\pi i} \frac{dw}{w-z} \qquad (0.4)$$

  - the Bochner-Martinelli integral for a Lipschitz domain $D \subset \mathbb{C}^n$ *is* bounded:

$$L^p(bD,\sigma) \to L^p(bD,\sigma), \quad 1 < p < \infty$$

  - the Bochner-Martinelli integral for e.g., a Lipschitz domain $D \subset \mathbb{C}^n$ does reproduce *holomorphic (i.e. analytic)* functions ("Cauchy formula").

A candidate for the Cauchy kernel in $\mathbb{C}^n$

The Bochner-Martinelli kernel:

$$H(w,z) = \frac{1}{(2\pi i)^n} \sum_{\ell=1}^{n} \frac{\overline{w}_\ell - \overline{z}_\ell}{|w-z|^{2n}}\, dw_\ell \bigwedge_{\nu\neq\ell} d\overline{w}_\nu \wedge dw_\nu \qquad (0.5)$$

- **An unfavorable feature of BM kernel**:
  - $n \geq 2 \Rightarrow$
    BM kernel (0.5) is **not holomorphic** as a function of $z \in D$
  - As a consequence, the BM integral does not produce holomorphic functions (from, say, $C^1(bD)$- data): this fact **limits the applicability of the BM integral to the study of problems in complex function theory**.

Objective

Extend the 1-dimensional theory for $\mathcal{C}$ to $\mathbb{C}^n$, $n \geq 1$:

- *Find* a higher-dimensional analog of the Cauchy kernel:

$$H(w,z) = \frac{1}{2\pi i}\frac{dw}{w-z}, \quad z \in D \subset \mathbb{C}, \quad w \in bD \subset \mathbb{C}$$

  which is now

  - meaningful when $z \in D \subset \mathbb{C}^n$, $w \in bD \subset \mathbb{C}^n$, $n \geq 1$
  - for D with "minimal" regularity
  - **and, holomorphic as a function of** $z \in D$

- Show that the operator defined via this new kernel is *bounded*:

$$L^p(bD,\sigma) \to L^p(bD,\sigma), \quad 1 < p < \infty$$

Effects of the requirement that the kernel be holomorphic

- Requirement on domain's geometry:

  D has to have some "convexity"

- Requirement on domain's regularity:

  D needs to be more regular than "Lipschitz"

Why "convexity" ????

- The L^p-theory of the (holomorphic) Cauchy integral for $D \Subset \mathbb{C}^n$, $n \geq 2$ requires dealing with
  - Dimension-induced obstructions ($\mathbb{C}^n$ vs. $\mathbb{C}$)
  - Complex-Structure-induced obstructions ($\mathbb{C}^n$ vs. $\mathbb{R}^{2n}$)
- These obstructions ultimately lead to the requirement that

$D \Subset \mathbb{C}^n$ be "**pseudoconvex**"

Why "Pseudoconvexity"?

"Pseudoconvexity" is a dimension-induced phenomenon:

When you look for a holomorphic kernel $H(w,z)$, what you are really looking for is *a function that is holomorphic in* $z \in D$ *and is singular at (any)* $w \in bD$ *(cannot be extended holom. past* w*)* i.e., D must be a maximal domain of analyticity (= "*domain of holomorphy*")

- In dimension 1 one can always find such a function (no matter what D looks like): just take

$$H(w,z) = \frac{1}{w-z}$$

- In dimension $n \geq 2$ there are examples of domains D_0 for which this may not be the case ("bad domains" – known since early '1900s!)
- Levi problem (connects "analysis" with "geometry"):

$D \subset \mathbb{C}^n$ is a domain of holomorphy $\iff$ D is "pseudoconvex"

Settings where things are known to work:

Henkin; Ramirez; Kerzman-Stein (1978):

$$D \in C^k, \quad k \geq 3 \quad \text{and} \quad \text{strongly Levi-pseudoconvex}$$

- Kernel: via an algebraic construct (*Cauchy-Fantappié theory*)
- Proof of $L^p(bD, \sigma) \to L^p(bD, \sigma)$-regularity:

  by way of "*osculation by model domain*":

$$\{z \mid \operatorname{Im} z_n > |z_1|^2 + \cdots + |z_{n-1}|^2\}$$

the Siegel Upper Half Space

Settings where things are known to work:

L. - Stein (2016)

$$D \in C^k, \quad k = 2 \quad \text{and} \quad \text{strongly Levi-pseudoconvex}$$

- Kernel(s): a family of Cauchy-Fantappiè terms
- Proof of $L^p \to L^p$-regularity: $T(1)$ theorem.
  (Original method (“osculation by model domain”) breaks down as soon as regularity of D is below the class C^3.)

Setting of current interest:

L. - Stein (2014):

$D \Subset \mathbb{C}^n, \quad D \in C^{1,1}$ and strongly $\mathbb{C}$-linearly convex, i.e.

- D has a defining function of class $C^{1,1}$, i.e.
  - $D = \{\rho < 0\}$ and $bD = \{w \mid \rho(w) = 0\}$, with
  - $\rho : \mathbb{C}^n \to \mathbb{R}, \qquad \rho \in C^1(\mathbb{C}^n)$
  - $\nabla\rho(w) \neq 0,\ w \in bD$, and $\nabla\rho \in \mathrm{Lip}(\mathbb{C}^n)$

and

- $d^E(z, w + T_w^{\mathbb{C}}) \ \geq \ c|w - z|^2$ if $z \in \overline{D}$ and $w \in bD$

Example: Siegel upper half space: $D = \{z \in \mathbb{C}^2 \mid \operatorname{Im} z_2 > |z_1|^2\}$ is strongly $\mathbb{C}$-linearly convex, but not strongly convex (because $\ell = \{(0 + i0, x_2 + i0) \mid x_2 \in \mathbb{R}\} \subset bD$)

Kernel

Cauchy-Leray kernel:

$$H(w,z)=\frac{1}{(2\pi i)^n}\frac{\partial\rho(w)\wedge(\bar\partial\partial\rho(w))^{n-1}}{\langle\partial\rho(w),w-z\rangle^n},\quad w\in bD,\quad z\in D$$

- ρ is (*any*) defining function for D
- $\partial\rho(w)=\sum_{j=1}^{n}\frac{\partial\rho}{\partial\zeta_j}(w)dw_j\,;\quad \bar\partial\partial\rho=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial\zeta_j\partial\bar\zeta_k}(w)dw_j\wedge d\overline{w}_k$
- $\langle\zeta,\eta\rangle:=\sum\zeta_j\,\eta_j,\quad \zeta,\eta\in\mathbb{C}^n$
- first introduced by J. Leray (1950s) in the setting of C^2-smooth, strongly convex domains D.
  Revisited by T. Hansson (1999) in the specialized context of a family of C^∞-smooth, weakly convex ellipsoids.

Cauchy-Leray kernel: good news

Suppose for the moment that D is of class C^2:

$$H(w,z) = \frac{1}{(2\pi i)^n}\frac{\partial\rho(w)\wedge(\overline{\partial}\partial\rho(w))^{n-1}}{\langle\partial\rho(w), w-z\rangle^n} \qquad (0.6)$$

- If $n = 1$ then Cauchy-Leray is the one-dim Cauchy kernel:

$$\frac{\partial\rho(w)}{\langle\partial\rho(w), w-z\rangle} = \frac{\rho'(w)dw}{\rho'(w)(w-z)} = \frac{dw}{w-z}$$

- $H(w,z)$ is holomorphic wrt $z\in D$ because denominator does not vanish **by strong $\mathbb{C}$-linear convexity**:

$$|\langle\partial\rho(w), w-z\rangle| \approx d^E(z, w+T_w^{\mathbb{C}}) \;\geq\; c|w-z|^2 > 0$$

Cauchy-Leray kernel: caveats

Suppose now D that is only of class $C^{1,1}$:

$$H(w,z) = \frac{1}{(2\pi i)^n}\frac{\partial\rho(w)\wedge(\bar\partial\partial\rho(w))^{n-1}}{\langle\partial\rho(w), w-z\rangle^n} \qquad (0.7)$$

- By Rademacher Theorem:

$$\rho \in C^{1,1}(\mathbb{C}^n) \quad \Rightarrow \quad \nabla^2\rho \in L^\infty(\mathbb{C}^n)$$

- in particular, $\nabla^2\rho(w)$ is defined only a.e. $w \in \mathbb{C}^n$
- but bD has measure 0 in $\mathbb{C}^n$
- so, $\nabla^2\rho$ may be undefined on bD. In particular

$$\bar\partial\partial\rho, \text{ and thus } H(w,z), \text{ may be undefined}$$

An example

For

$$F : \mathbb{C} \to \mathbb{R} \quad \text{given by} \quad F(x + iy) := |x|$$

and

$$D := \{x + iy \,|\, x < 0\} \subset \mathbb{C}$$

we have

- D is a smooth domain in $\mathbb{C}$;
- $F \in Lip(\mathbb{C})$ and so $\nabla F \in L^\infty(\mathbb{C})$

However, ∇F is undefined on $bD = \{x + iy \,|\, x = 0\}$

An example

For

$$F : \mathbb{C} \to \mathbb{R} \quad \text{given by} \quad F(x+iy) := |x|$$

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we have

- D is a smooth domain in $\mathbb{C}$;
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However, ∇F is undefined on $bD = \{x + iy \,|\, x = 0\}$

On the other hand, *j^*dF is well-defined on bD (in fact, $j^*dF = dj^*F \equiv 0$)*

$$j : bD \hookrightarrow \mathbb{C}$$

Cauchy-Leray kernel: the role of tangential components

$$H(w,z) = \frac{1}{(2\pi i)^n}\, j^*\left(\frac{\partial\rho(w)\wedge(\overline{\partial}\partial\rho(w))^{n-1}}{\langle\partial\rho(w), w-z\rangle^n}\right)$$

Proposition (L. – Stein)

Suppose $F \in C^{1,1}(\mathbb{C}^n)$ (with $n \geq 2$) and $D \subset \mathbb{C}^n$ is of class $C^{1,1}$. Then there exists a (unique) 2-form on bD, which we write as $j^*(\overline{\partial}\partial F)$, whose coefficients are in $L^\infty(bD)$ and satisfies

$$\int\limits_{bD} j^*(\overline{\partial}\partial F)\wedge\psi = \int\limits_{bD} j^*(\partial F)\wedge d(\psi)$$

for all $(2n-3)$-forms ψ on bD that are of class C^1.

Outcomes:

- Cauchy-Leray kernel is well-defined (meaningful):
- $H(w,z)$ reproduces holomorphic functions (*"Cauchy formula"*) and is "canonical" (*independent of choice of ρ*).

Cauchy-Leray integral: main result

Theorem

Suppose $D \subset \mathbb{C}^n$ is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$. Then, the Cauchy-Leray integral:

$$f \mapsto \mathcal{C}(f)(z) := \frac{1}{(2\pi i)^n} \int\limits_{w \in bD} f(w)\, j^* \left(\frac{(\partial\rho(w) \wedge (\overline{\partial}\partial\rho(w))^{n-1})}{\langle \partial\rho(w), w - z \rangle^n} \right)$$

initially defined for functions in $C^1(bD)$, extends to a bounded linear operator:

$$L^p(bD, \lambda) \to L^p(bD, \lambda), \quad 1 < p < \infty$$

where λ is the Leray-Levi measure

$$d\lambda(w) := j^*(\partial\rho(w) \wedge (\overline{\partial}\partial\rho(w))^{n-1})$$

Cauchy-Leray integral: L^p-regularity

Proof of $L^p \to L^p$-regularity: goes by way of

- $T(1)$-Theorem in the special case:

$$T(1) = 0; \quad T^*(1) = 0$$

- for a space of homogeneous type informed by the geometry and regularity of the ambient domain D.

A space of homogeneous type that works for us

Theorem

Suppose $D \Subset \mathbb{C}^n$ is strongly $\mathbb{C}$-linearly convex and of class $C^{1,1}$.

Then, $(X, \mathtt{d}, \lambda)$ is a space of homogeneous type, with:

- *Set:* $X := bD$
- *Quasimetric:* $\mathtt{d}(w, z) := |\langle \partial\rho(w), w - z\rangle|^{1/2}, \quad w, z \in bD$
- *Doubling measure: Leray-Levi meas.:* $d\lambda = j^*(\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1})$

(in fact: $\lambda(\{w \in bD,\ \mathtt{d}(w, z) < r\}) \approx r^{2n}$*)*

Note: *Leray-Levi measure λ plays a distinguished role which is akin to harmonic measure for Laplace operator....*

A few words about the proof

A key ingredient in the proof of the weak-boundedness property and of the cancellation conditions:

$$T(1) = 0; \quad T^*(1) = 0$$

are two basic identities that in effect express the Cauchy-Leray kernel and its adjoint kernel as *appropriate derivatives*. Namely:

$$H(w,z) = d_w\omega(w,z) + \mathfrak{r}(w,z), \quad \text{and}$$

$$\overline{H(z,w)} = d_w\widetilde{\omega}(w,z) + \widetilde{\mathfrak{r}}(w,z), \quad \text{where}$$

- the coefficients of ω (resp. $\widetilde{\omega}$) are absolutely integrable and have better homogeneity than $H(z,w)$ (resp. $\overline{H(w,z)}$), i.e.

  they have $\langle \partial\rho(w), w - z\rangle^{-n+1}$ vs. $\langle \partial\rho(w), w - z\rangle^{-n}$
- the remainders $\mathfrak{r}$ and $\widetilde{\mathfrak{r}}$ have sufficient integrability to ensure that the corresponding integral operators map:

  $$C(bD) \mapsto C(\overline{D})$$
- From this it follows that $\mathcal{C}$ is weakly bounded and also that

  $\mathfrak{h} := \mathcal{C}^*(1) \in C(\overline{D})$ (in fact $|\mathfrak{h}(w) - \mathfrak{h}(z)| \lesssim \mathfrak{d}(w,z)^\alpha$, $0 < \alpha < 1$).

Comparison with proof for 1-dimensional setting

Remarkably the “basic identities” are meaningful only for $n > 1$, because a one-dimensional analogue would necessarily involve a logarithmic term, invalidating their use: i.e., for $n = 1$ one has:

$$H(w,z) = \frac{dw}{w-z} = d_w\omega(w,z) + \mathfrak{r}(w,z)$$

with

- $\omega(w,z) := \log(w-z)$
- $\mathfrak{r}(w,z) = 0$

but $\log(w-z)$ does not have the appropriate homogeneity that would automatically ensure the weak boundedness property.

Further results

- $\mathcal{C}$ is also bounded: $L^p(bD, \sigma) \to L^p(bD, \sigma)$ (σ=Induced Lebesgue meas.) because $\lambda \approx \sigma$ as a consequence of
  - Strong $\mathbb{C}$-linear convexity of D ("$\lambda \gtrsim \sigma$")
  - $C^{1,1}$-regularity of D ("$\lambda \lesssim \sigma$")

- L.-Stein (2017): Strong $\mathbb{C}$-lin. convexity is optimal:

$$D := \{x_1^2 + y_1^4 + x_2^2 + (y_2 - 1)^2 < 1\}$$

  - D is smooth and strictly (but not strongly) convex
  - $\mathcal{C}$ unbounded in L^p for all $1 < p < \infty$.

- L.-Stein (2017): $C^{1,1}$ category also optimal:

$$D_\alpha := \{|x_1|^{1+\alpha} + y_1^2 + x_2^2 + (y_2 - 1)^2 < 1\}, \ 0 < \alpha < 1.$$

  - D_α strongly convex and of class $C^{1,\alpha}$ (but not $C^{1,1}$)
  - $\mathcal{C}$ unbounded in L^p for all $1 < p < \infty$.

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Thank You!