Jump inequalities for translation-invariant polynomial averages and singular integrals on $\mathbb{Z}^d$

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Maximal Radon transform

The maximal Radon transform is defined for $x \in \mathbb{R}^d$ by setting

$$\mathcal{M}_*^{\mathcal{P}} f(x) = \sup_{t>0} \left|\mathcal{M}_t^{\mathcal{P}} f(x)\right|,$$

where

$$\mathcal{M}_t^{\mathcal{P}} f(x) = \frac{1}{|B_t|} \int_{B_t} f(x - \mathcal{P}(y)) dy,$$

$B_t = \{y \in \mathbb{R}^k : |y| < t\}$ and

$$\mathcal{P}(y) = (\mathcal{P}_1(y), \ldots, \mathcal{P}_d(y))$$

is a polynomial mapping, i.e. $\mathcal{P}_j(y)$ is a real-valued polynomial on $\mathbb{R}^k$.

- It is very well known that for every $p > 1$ there is a $C_p > 0$ such that

  $$\|\mathcal{M}_*^{\mathcal{P}} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

  for any $f \in L^p(\mathbb{R}^d)$.

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L^2 maximal estimates along the parabola

Consider the maximal function $\mathcal{M}_* f(x_1, x_2)$ along the parabola, i.e. corresponding with the averages

$$\mathcal{M}_r f(x_1, x_2) = \frac{1}{2r} \int_{-r}^{r} f(x_1 - y, x_2 - y^2) dt.$$

Let Φ be a smooth compactly supported function such that $\int_{\mathbb{R}^2} \Phi(y) dy = 1$ and define $\Phi_n(x_1, x_2) = 2^{-3n} \Phi(2^{-n} x_1, 2^{-2n} x_2)$. Then it is easy to see that

$$\left\| \mathcal{M}_* f \right\|_{L^2} \leq \left\| \sup_{n \in \mathbb{Z}} |\Phi_n * f| \right\|_{L^2} + \left\| \left(\sum_{n \in \mathbb{Z}} \left| \mathcal{M}_{2^n} f - \Phi_n * f \right|^2 \right)^{1/2} \right\|_{L^2} \lesssim \|f\|_{L^2}$$

$$+ \left(\sum_{n \in \mathbb{Z}} \left\| \mathcal{M}_{2^n} f - \Phi_n * f \right\|_{L^2}^2 \right)^{1/2} \lesssim \|f\|_{L^2} + \sup_{\xi, \eta \in \mathbb{R}} \left(\sum_{n \in \mathbb{Z}} \left| \mathfrak{m}_{2^n}(\xi, \eta) - \widehat{\Phi_n}(\xi, \eta) \right|^2 \right)^{1/2}$$

where $\mathfrak{m}_{2^n}(\xi, \eta) = \frac{1}{2} \int_{-1}^{1} e^{-2\pi i (\xi 2^n y + \eta (2^n y)^2)} dy$ and

$$\sum_{n \in \mathbb{Z}} \left| \mathfrak{m}_{2^n}(\xi, \eta) - \widehat{\Phi}(2^n \xi, 2^{2n} \eta) \right| \lesssim \sum_{n \in \mathbb{Z}} \min \left\{ (|2^n \xi| + |2^{2n} \eta|), (|2^n \xi| + |2^{2n} \eta|)^{-1/2} \right\} \lesssim 1$$

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L^p maximal estimates along the parabola

Let us consider a partition of unity $(\phi_n : n \in \mathbb{Z})$ such that

$$\sum_{n \in \mathbb{Z}} \phi_n(\xi) = 1 \quad \text{for every} \quad \xi \in \mathbb{R}^2 \setminus \{0\}$$

where $\operatorname{supp} \phi_n \subseteq \{(\xi, \eta) \in \mathbb{R}^2 : |2^n \xi| + |2^{2n} \eta| \simeq 1\}$ and

$$f = \sum_{k \in \mathbb{Z}} \mathcal{F}^{-1}(\phi_n \widehat{f}).$$

Now it suffices to show that for every $p > 1$

$$\sum_{k \in \mathbb{Z}} \Big\| \Big(\sum_{n \in \mathbb{Z}} \big| \mathcal{M}_{2^n} \big(\mathcal{F}^{-1}(\phi_{n+k} \widehat{f}\,)\big) - \Phi_n * \big(\mathcal{F}^{-1}(\phi_{n+k} \widehat{f}\,)\big) \big|^2 \Big)^{1/2} \Big\|_{L^p} \lesssim \|f\|_{L^p}.$$

Indeed, for $p > 1$ and each $k \in \mathbb{Z}$ by the Littlewood–Paley theory we have

$$\Big\| \Big(\sum_{n \in \mathbb{Z}} \big| \mathcal{F}^{-1}\big((\mathfrak{m}_{2^n} - \widehat{\Phi}_n) \phi_{n+k} \widehat{f}\big) \big|^2 \Big)^{1/2} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{n \in \mathbb{Z}} \big| \mathcal{F}^{-1}(\phi_{n+k} \widehat{f}) \big|^2 \Big)^{1/2} \Big\|_{L^p} \lesssim \|f\|_{L^p}.$$

For $p = 2$ there is $\delta > 0$ such that for each $k \in \mathbb{Z}$ we have

$$\Big\| \Big(\sum_{n \in \mathbb{Z}} \big| \mathcal{F}^{-1}\big((\mathfrak{m}_{2^n} - \widehat{\Phi}_n) \phi_{n+k} \widehat{f}\,\big) \big|^2 \Big)^{1/2} \Big\|_{L^2} \lesssim 2^{-\delta |k|} \|f\|_{L^2}.$$

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Bourgain's ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with an invertible measure-preserving transformation $T : X \to X$.

In the mid 1980's Bourgain extended Birkhoff's ergodic theorem and showed that for every $f \in L^p(X, \mu)$ with $p > 1$ there is a function $f^* \in L^p(X, \mu)$ such that

$$\lim_{N\to\infty} A_N f(x) = f^*(x)$$

μ-almost everywhere on X for the averages

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Pointwise convergence

Although, for Birkhoff's averaging operator, it was not very difficult to find a dense class of functions (say on $L^2(X,\mu)$) for which pointwise convergence holds, for Bourgain's averaging operator

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along the polynomials P of degree > 1, it is a hard problem. Even for $P(n) = n^2$, since $(n+1)^2 - n^2 = 2n + 1$.

For overcoming the lack of dense class, Bourgain showed

- L^p boundedness of the maximal function,
- Given a lacunary sequence $(N_j : j \in \mathbb{N})$, for each $J > 0$ there is $C > 0$ such that

$$\Big(\sum_{j=0}^{J} \Big\| \sup_{N \in [N_j, N_{j+1})} \big|A_N^P f - A_{N_j}^P f\big| \Big\|_{L^2}^2\Big)^{1/2} \leq C J^c \|f\|_{L^2}$$

for some $c < 1/2$.

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Variational seminorm

For any complex-valued functions $(a_t(x) : t > 0)$ and $r \geq 1$ the variational seminorm is

$$V_r(a_t(x) : t > 0) = \sup_{\substack{t_0 < t_1 < \ldots < t_J \\ t_j > 0}} \left(\sum_{j=0}^{J-1} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r}.$$

Observe that

- $V_r(a_t(x) : t > 0) < \infty$ implies $(a_t(x) : t > 0)$ is a Cauchy sequence.
- Moreover, we have

$$\sup_{t>0} |a_t(x)| \leq V_r(a_t(x) : t > 0) + |a_{t_0}(x)|$$

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For any complex-valued functions $(a_t(x) : t > 0)$ and $r \geq 1$ the variational seminorm is

$$V_r(a_t(x) : t > 0) = \sup_{\substack{t_0 < t_1 < \ldots < t_J \\ t_j > 0}} \left(\sum_{j=0}^{J-1} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r}.$$

Observe that

- $V_r(a_t(x) : t > 0) < \infty$ implies $(a_t(x) : t > 0)$ is a Cauchy sequence.
- Moreover, we have

$$\sup_{t>0} |a_t(x)| \leq V_r(a_t(x) : t > 0) + |a_{t_0}(x)|$$

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In fact there is a simpler object to control r-variations.

Jump function

For any complex-valued functions $(a_t(x) : t > 0)$ and any $\lambda > 0$ we define λ-jump function

$$N_\lambda(a_t(x) : t > 0) = \sup\big\{J \in \mathbb{N}_0 : \exists_{\,0<t_1<\ldots<t_J} \min_{1\leq j<J} |a_{t_{j+1}}(x) - a_{t_j}(x)| > \lambda\big\}.$$

- The jumps $N_\lambda(a_t(x) : t > 0)$ are pointwisely comparable with the r-variation. Namely we have a uniform in $\lambda > 0$ bound

$$\lambda\big[N_\lambda(a_t(x) : t > 0)\big]^{1/r} \leq V_r(a_t(x) : t > 0).$$

- The advantage of $N_\lambda(a_t(x) : t > 0)$ is that we have a reverse inequality in the following sense:

Lemma

Let $1 \leq p \leq \infty$ and $1 \leq \rho < r \leq \infty$ then

$$\big\|V_r(a_t : t > 0)\big\|_{L^p} \lesssim_{p,\rho} \left(\frac{r}{r-\rho}\right)^{\max\{1/p,1/\rho\}} \sup_{\lambda>0} \big\|\lambda\big[N_\lambda(a_t(x) : t > 0)\big]^{1/\rho}\big\|_{L^p}.$$

This inequality can be extended to $L^{p,\infty}$ spaces as well.

Variational estimates in the continuous setup

Let $B_t = \{y \in \mathbb{R}^k : |y| < t\}$ and recall that

$$\mathcal{M}_t^{\mathcal{P}} f(x) = \frac{1}{|B_t|} \int_{B_t} f(x - \mathcal{P}(y)) dy,$$

where $\mathcal{P} : \mathbb{R}^k \to \mathbb{R}^d$ is a polynomial mapping.

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Theorem (Jones, Seeger and Wright)

For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_p > 0$ such that for all $f \in L^p(\mathbb{R}^d)$

$$\left\| V_r\big(\mathcal{M}_t^{\mathcal{P}} f : t > 0\big) \right\|_{L^p} \leq C_p \frac{r}{r-2} \|f\|_{L^p}.$$

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For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in L^p(\mathbb{R}^d)$

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The strategy in Jones, Seeger and Wright's proof

We have for any $r \geq 2$ that

$$V_r\big(\mathcal{M}_t^{\mathcal{P}} f : t > 0\big) \lesssim_r V_r\big(\mathcal{M}_{2^n}^{\mathcal{P}} f : n \in \mathbb{Z}\big) + \Big(\sum_{n\in\mathbb{Z}} V_2\big(\mathcal{M}_t^{\mathcal{P}} f : t \in [2^n, 2^{n+1})\big)^2\Big)^{1/2}.$$

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Martingales inequalities

One of the main ingredients in the proof was Lépingle's inequality which says that for a general bounded martingale $(\mathfrak{f}_n : n \in \mathbb{Z})$ we have

Theorem (Lépingle)

For every $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_p > 0$ such that

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Moreover, at the endpoint for $p = 1$ we have weak-type $(1, 1)$ inequality.

- However, for $r = 2$ we have that

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Long variations and jumps

Now we apply the inequalities from the last display for dyadic matringales $(\mathfrak{f}_n : n \in \mathbb{Z})$ taken with respect to Christ's cubes which correspond to the nonisotropic dilations determined by the underlying polynomial mapping $\mathcal{P}$.

- Then we have

$$V_r\big(\mathcal{M}_{2^n}^{\mathcal{P}} f - \mathfrak{f}_n : n \in \mathbb{Z}\big) + \lambda\big[N_\lambda\big(\mathcal{M}_{2^n}^{\mathcal{P}} f - \mathfrak{f}_n : n \in \mathbb{Z}\big)\big]^{1/2} \lesssim \Big(\sum_{n\in\mathbb{Z}} \big|\mathcal{M}_{2^n}^{\mathcal{P}} f - \mathfrak{f}_n\big|^2\Big)^{1/2}$$

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Variational estimates in the discrete setup

Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d) : \mathbb{Z}^k \to \mathbb{Z}^d$ be a polynomial mapping with integer coefficients. Define Radon averages

$$M_N^{\mathcal{P}} f(x) = \frac{1}{|\mathbb{B}_N|} \sum_{y \in \mathbb{B}_N} f(x - \mathcal{P}(y)),$$

where $\mathbb{B}_N = \{y \in \mathbb{Z}^k : |y| \leq N\}$. Then

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For every $p \in (1, \infty)$ there is $C_p > 0$ such that for all $f \in \ell^p\big(\mathbb{Z}^d\big)$

$$\sup_{\lambda > 0} \big\| \lambda \big[N_\lambda \big(M_N^{\mathcal{P}} f : N \in \mathbb{N} \big) \big]^{1/2} \big\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Moreover, the constant C_p is independent of coefficients of the polynomial mapping $\mathcal{P}$.

Jump estimates for truncated Radon transform

Suppose that $K \in \mathcal{C}^1\big(\mathbb{R}^k \setminus \{0\}\big)$ is a Calderón–Zygmund kernel obeying

$$|y|^k|K(y)| + |y|^{k+1}|\nabla K(y)| \leq 1$$

for all $y \in \mathbb{R}^k \setminus \{0\}$ and a cancellation condition

$$\int_{\lambda_1 \leq |y| \leq \lambda_2} K(y)\mathrm{d}y = 0$$

for all $\lambda_1 < \lambda_2$. Define truncated Radon transform

$$T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f\big(x - \mathcal{P}(y)\big)K(y)$$

where $\mathbb{B}_N = \{x \in \mathbb{Z}^k : |x| \leq N\}$.

Theorem (M., E.M. Stein and P. Zorin–Kranich)

For every $1 < p < \infty$ there is $C_p > 0$ such that for all $f \in \ell^p\big(\mathbb{Z}^d\big)$

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Proof of the variational estimates

To simplify arguments let us consider that $\mathcal{P}(x) = x^d$ and $d \geq 2$. We prove that for any $r > 2$

$$\left\| V_r(M_{2^n}^{\mathcal{P}} f : n \in \mathbb{N}_0) \right\|_{\ell^2(\mathbb{Z})} \leq C \frac{r}{r-2} \|f\|_{\ell^2(\mathbb{Z})}.$$

Let

$$K_N(x) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\mathcal{P}(k)}(x),$$

then

$$M_N^{\mathcal{P}} f(x) = K_N * f(x).$$

For $f \in \ell^1(\mathbb{Z})$ let

$$\widehat{f}(\xi) = \sum_{k \in \mathbb{Z}} e^{2\pi i \xi k} f(k)$$

and observe that

$$m_N(\xi) = \widehat{K}_N(\xi) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \xi k^d} \quad (\xi \in \mathbb{T}).$$

Some heuristics

- First of all we have to understand the behaviour of

$$m_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \xi k^d}.$$

- We see that if ξ is an integer, then

$$m_N(\xi) = 1. \qquad \text{There is no decay at infinity like } \lesssim |N\xi|^{-1/d}!$$

- Now we would like to replace $m_N(\xi)$ with the integral

$$\Phi_N(\xi) = \int_0^1 e^{2\pi i \xi (Nx)^d}\, dx.$$

- However, we can not do this naively, since the derivative of the phase function $k^d\xi$ arising in the exponential sum is equal to $dk^{d-1}\xi$ and may be large. Thus in general we have no control over the error term

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Gaussian sums

If $\xi = a/q$ and $(a,q) = 1$ then we see that $m_N(a/q)$ behaves like a complete Gaussian sum

$$G(a/q) = \frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}.$$

Indeed,

$$m_N(a/q) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{a}{q} k^d} = \frac{1}{N}\sum_{r=1}^{q} \sum_{-\frac{r}{q} < k \le \frac{N-r}{q}} e^{2\pi i \frac{a}{q} (qk+r)^d} \simeq \frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}.$$

This suggests that the asymptotics for m_N should be concentrated in some neighbourhoods of Diophantine approximations of ξ with small denominators.

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Small denominators - asymptotic formula for $m_N(\xi)$

From Dirichlet's principle for any $\xi \in [0,1]$ and we can always find $a/q \in [0,1)$ such that $1 \le q \le N^d (\log N)^{-\beta}$, $(a,q) = 1$ and

$$\left|\xi - \frac{a}{q}\right| \le \frac{(\log N)^\beta}{qN^d}$$

for any $\beta > 0$. If $1 \le q \le (\log N)^\beta$

$$m_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \xi \cdot k^d} = \frac{1}{N}\sum_{r=1}^{q} \sum_{-\frac{r}{q} < n \le \frac{N-r}{q}} e^{2\pi i (\xi - \frac{a}{q})(qn+r)^d} e^{2\pi i \frac{a}{q}(qn+r)^d}$$

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Therefore, if ξ is in the neighbourhood of a/q as above, we have

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Large denominators - Weyl's inequality

It was observed by Hardy and Littlewood that if $|\xi - a/q| \le \frac{(\log N)^\beta}{qN^d} \le q^{-2}$ and $(a,q) = 1$ and $(\log N)^\beta \le q \le N^d (\log N)^{-\beta}$ then

$$|m_N(\xi)| = \left| \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \xi k^d} \right| \lesssim (\log N)^{-\alpha}$$

for any $\alpha > \alpha_\beta$. This follows from the following variant of Weyl's inequality.

Lemma (Weyl's inequality)

Let $P(x) = a_d x^d + \ldots + a_1 x$. Suppose there are $(a,q) = 1$ such that $|a_d - a/q| \le q^{-2}$. Then there is $C > 0$ such that

$$\frac{1}{N} \left| \sum_{m=1}^{N} e^{2\pi i P(m)} \right| \le C \log N \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^d} \right)^{1/2^{d-1}}$$

uniformly in N and q.

Weyl's inequality is usually formulated with N^ε loss instead of $\log N$. However for our purposes we need a more subtle variant with logarithmic loss.

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Projections $\Xi_{n^l}(\xi)$

For an integer $l \in \mathbb{N}$ and $\chi > 0$ let us define the following projections

$$\Xi_{n^l}(\xi) = \sum_{a/q \in \mathscr{U}_{n^l}} \eta(2^{n(d-\chi)}(\xi - a/q))$$

with a smooth cuf-off function η and

$$\mathscr{U}_{n^l} = \{a/q \in \mathbb{T} : \ (a, q) = 1 \text{ and } q \in \mathbf{P}_{n^l}\},$$

where the denominators $q \in \mathbf{P}_{n^l}$ have appropriate limitation in terms of their prime power factorization.

Since

$$m_{2^n}(\xi) = m_{2^n}(\xi)(1 - \Xi_{n^l}(\xi)) + m_{2^n}(\xi)\Xi_{n^l}(\xi),$$

the first term is supported in the regime where Weyl's inequality is efficient. The second we approximate by the integral.

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The highly oscillatory part $m_{2^n}(1 - \Xi_{n^l})$

Form Weyl's inequality we have

$$|m_{2^n}(\xi)| = \left| \frac{1}{2^n} \sum_{k=1}^{2^n} e^{2\pi i \xi k^d} \right| \lesssim (n+1)^{-\alpha}$$

for a large $\alpha > 0$, provided that $1 - \Xi_{n^l}(\xi) \neq 0$. Therefore, by Plancherel's theorem

$$\begin{aligned}
\big\| V_r\big(\mathcal{F}^{-1}\big(m_{2^n}(1 - \Xi_{n^l})\hat{f}\big) : n \in \mathbb{N}_0\big)\big\|_{\ell^2} &\leq \big\| V_1\big(\mathcal{F}^{-1}\big(m_{2^n}(1 - \Xi_{n^l})\hat{f}\big) : n \in \mathbb{N}_0\big)\big\|_{\ell^2} \\
&\leq \sum_{n \in \mathbb{N}_0} \big\| \mathcal{F}^{-1}\big(m_{2^n}(1 - \Xi_{n^l})\hat{f}\big)\big\|_{\ell^2} \\
&\lesssim \sum_{n \in \mathbb{N}_0} (n+1)^{-2} \|f\|_{\ell^2} \lesssim \|f\|_{\ell^2}.
\end{aligned}$$

The asymptotic part $m_{2^n}\Xi_{n^l}$

Recall that if $a/q \in \mathscr{U}_{n^l}$ then we have

$$m_{2^n}(\xi) \simeq G(a/q)\cdot\Phi_{2^n}(\xi - a/q) = \left(\frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}\right)\cdot\left(\int_0^1 e^{2\pi i(\xi-\frac{a}{q})(2^n x)^d}dx\right).$$

Therefore,

$$m_{2^n}(\xi)\Xi_n(\xi) \simeq \sum_{s\geq 0} m_{2^n}^s(\xi)$$

where

$$m_{2^n}^s(\xi) = \sum_{a/q\in\mathscr{W}_{s^l}} G(a/q)\Phi_{2^n}(\xi - a/q)\eta\big(2^{s(d-\chi)}(\xi - a/q)\big),$$

with $\mathscr{W}_{s^l} \subseteq \mathscr{U}_{n^l}$ and has the property that if $q \in \mathscr{W}_{s^l}$ then $q \geq s^l$.

The task now is to show that for any $s \geq 0$ we have

$$\left\|V_r\left(\mathcal{F}^{-1}\left(m_{2^n}^s \hat{f}\right) : n \in \mathbb{N}_0\right)\right\|_{\ell^2} \leq C(s+1)^{-\delta l+1}\|f\|_{\ell^2}$$

for every $f \in \ell^2(\mathbb{Z})$, where $\delta > 0$ comes from the estimate $|G(a/q)| \leq Cq^{-\delta}$.

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The case $0 \leq n \leq 2^s$

Simple numerical inequality

For any sequence $(a_j : 0 \leq j \leq 2^s) \subseteq \mathbb{C}$, for $s \in \mathbb{N} \cup \{0\}$ and $r > 2$, we have

$$V_r(a_n : 0 \leq n \leq 2^s) \leq \sqrt{2} \sum_{i=0}^{s} \Big(\sum_{j=0}^{2^{s-i}-1} |a_{(j+1)2^i} - a_{j2^i}|^2 \Big)^{1/2}$$

Hence by Plancherel's theorem we obtain

$$\begin{aligned}&\big\| V_r\big(\mathcal{F}^{-1}\big(m_{2^n}^s \hat{f}\big) : 0 \leq n \leq 2^s\big)\big\|_{\ell^2} \\ &\quad\lesssim \Big\| \sum_{i=0}^{s} \Big(\sum_{j=0}^{2^{s-i}-1} \Big(\sum_{k=j2^i}^{(j+1)2^i-1} \mathcal{F}^{-1}\big((m_{2^{k+1}}^s - m_{2^k}^s)\hat{f}\big)\Big)^2\Big)^{1/2}\Big\|_{\ell^2} \\ &\quad\lesssim \sum_{i=0}^{s} \Big(\sum_{j=0}^{2^{s-i}-1} \Big\| \sum_{k=j2^i}^{(j+1)2^i-1} (m_{2^{k+1}}^s - m_{2^k}^s)\hat{f}\Big\|_{L^2}^2 \Big)^{1/2} \lesssim s(s+1)^{-\delta l} \|f\|_{\ell^2}.\end{aligned}$$

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The case $n \geq 2^s$

For the second part we show that

$$\left\| V_r(\mathcal{F}^{-1}\left(m^s_{2^n}\hat{f}\right) : n \geq 2^s) \right\|_{\ell^2(\mathbb{Z})}$$
$$\lesssim (s+1)^{-\delta l+1} \sup_{\|g\|_{L^2(\mathbb{R})}=1} \left\| V_r(\mathcal{F}^{-1}\left(\Phi_N\right) * g : N \in \mathbb{N}) \right\|_{L^2(\mathbb{R})} \|f\|_{\ell^2(\mathbb{Z})}$$

which by Jones, Seeger and Wright theorem one can conclude that for any $r > 2$ and $p \in (1, \infty)$

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Where are the difficulties?

One of the major obstacle in the discrete theory is the following inequality

$$N_\lambda(F_t + G_t : t > 0)(x) \le N_{\lambda/2}(F_t : t > 0)(x) + N_{\lambda/2}(G_t : t > 0)(x).$$

Therefore, with this definition of jumps we cannot justify that

$$\begin{aligned}&\big\|\lambda\big[N_\lambda(\mathcal{F}^{-1}\big(m^s_{2^n}\hat{f}\big) : n \ge 2^s)\big]^{1/2}\big\|_{\ell^2(\mathbb{Z})}\\ &\lesssim (s+1)^{-\delta l+1} \sup_{\|g\|_{L^2(\mathbb{R})}=1} \big\|\lambda\big[N_\lambda(\mathcal{F}^{-1}\big(\Phi_N\big)*g : N \in \mathbb{N})\big]^{1/2}\big\|_{L^2(\mathbb{R})}\|f\|_{\ell^2(\mathbb{Z})}.\end{aligned}$$

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Real interpolation K-method

- Let (A_0, A_1) be a compatible couple of normed vector spaces (this means that they are both contained in some ambient topological vector space and the intersection $A_0 \cap A_1$ is dense both in A_0 and in A_1).
- For $a \in A_0 + A_1$ the K-functional is defined

$$K(t, a, A_1, A_2) = \inf_{a = a_1 + a_2} \left(\|a_0\|_{A_0} + t\|a_1\|_{A_1} \right)$$

- For $\theta \in (0,1)$ and $1 \leq r \leq \infty$ we define real interpolation space

$$[A_0, A_1]_{\theta,r} = \left\{ a \in A_0 + A_1 : \int_0^\infty \left(t^{-\theta} K(t, a, A_1, A_2) \right)^r \frac{dt}{t} < \infty \right\}.$$

- $[A_0, A_1]_{\theta,r}$ is equipped with the norm

$$\|a\|_{[A_0,A_1]_{\theta,r}} = \left(\int_0^\infty \left(t^{-\theta} K(t, a, A_1, A_2) \right)^r \frac{dt}{t} \right)^{1/r}.$$

- If $r = \infty$ we have $\|a\|_{[A_0,A_1]_{\theta,r}} = \sup_{t>0} t^{-\theta} K(t, a, A_1, A_2)$.

Example

- Let $(X, \mathcal{B}, \mu)$ be a measure space. For any measurable function $f : X \to \mathbb{C}$ we define its decreasing rearrangement by setting

$$f^*(t) = \inf\big\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda) \leq t\big\}.$$

- The Lorentz space $L^{p,q}(X, \mu)$ for $0 < p, q < \infty$ is defined as a space of those measurable functions $f : X \to \mathbb{C}$ for which

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t}\right)^{1/q} < \infty$$

and for $p = q$ we have $L^{p,q}(X, \mu) = L^p(X, \mu)$.

- For $q = \infty$ we have weak L^p space and

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t).$$

Example

- If $f \in L^1(X,\mu) + L^\infty(X,\mu)$ then

$$K(t,f,L^1,L^\infty) = \int_0^t f^*(t)dt.$$

- Consequently for $0 < p < \infty$ and $1 \leq q \leq \infty$

$$[L^1, L^\infty]_{\theta,q} = L^{p,q}(X,\mu),$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}.$$

Real interpolation for the jump function

Theorem

Let $(X, \mathcal{B}, \mu)$ *be a measure space. Then for every* $0 < p < \infty$ *and* $0 < q \leq \infty$ *there are constants* $0 < c_{p,q} \leq C_{p,q}$ *such that for every measurable function* $f : (0, \infty) \times X \to \mathbb{C}$ *we have*

$$c_{p,q} \sup_{\lambda>0} \left\| \lambda \left[N_\lambda(f(t,x) : t > 0) \right]^{1/2} \right\|_{L^{p,q}(X, d\mu(x))} \leq \left[L^\infty(V_\infty), L^{p/2,q/2}(V_1) \right]_{1/2,\infty}(f) \leq C_{p,q} \sup_{\lambda>0} \left\| \lambda \left[N_\lambda(f(t,x) : t > 0) \right]^{1/2} \right\|_{L^{p,q}(X, d\mu(x))}.$$

Therefore, if $1 < p = q < \infty$ the space

$$\left[L^\infty(V_\infty), L^{p/2,q/2}(V_1) \right]_{1/2,\infty}$$

is a Banach space.

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$$c_{p,q} \sup_{\lambda > 0} \left\| \lambda \big[N_\lambda (f(t,x) : t > 0) \big]^{1/2} \right\|_{L^{p,q}(X, d\mu(x))}$$
$$\leq \big[L^\infty(V_\infty), L^{p/2,q/2}(V_1) \big]_{1/2,\infty}(f)$$
$$\leq C_{p,q} \sup_{\lambda > 0} \left\| \lambda \big[N_\lambda (f(t,x) : t > 0) \big]^{1/2} \right\|_{L^{p,q}(X, d\mu(x))}.$$

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Thank You!