Geometric Averages in Harmonic Analysis

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Geometric Radon Operators

Fundamental Objects

- X_L – A subset of $\mathbb{R}^n$ or a manifold; comes with a measure.
- X_R – A subset of $\mathbb{R}^m$ or a manifold, also equipped with measure. Usually $X_L = X_R$ but not always.
- $\{\Sigma_{x_R}\}_{x_R \in X_R}$ – Some smoothly-varying family of submanifolds of X_L parametrized by points of X_R; equipped with measure.
- T – An averaging operator sending *a priori* continuous functions on X_L to continuous functions on X_R by integrating:

$$Tf(x_R) := \int_{\Sigma_{x_R}} f \; d\mu_{\Sigma_{x_R}}.$$

Basic Question

For which pairs $(\frac{1}{p}, \frac{1}{q})$ is there a finite constant C such that

$$||Tf||_{L^q(X_R)} \leq C||f||_{L^p(X_L)} \text{ for all } f \in C(X_R)?$$

Background and Basic Information

- Famous examples of geometric averaging operators are the X-ray transform and spherical averages.
- **Relating to Singular Integrals:** Fabes (1966); Stein and Wainger (1970); Nagel, Riviere, and Wainger (1974, 1976); Müller (1984, 1985); Christ (1985); Ricci and Stein (1988); Phong and Stein (1991, 1993); Christ, Nagel, Stein, and Wainger (1999); Stein and Street (2011, 2012)
- **Relating to FIOs/Oscillatory Integrals:** Greenleaf and Seeger (1994, 1998, 1999); Ricci (1997); Seeger (1998); Comech and Cuccagna (2003)
- **Combinatorial Approaches:** Littman (1971); Fefferman (1970); Zygmund (1974); Oberlin and Stein (1982); D. Oberlin (1987, 1997, 1999); Drury (1983, 1984, 1990); Christ (1984, 1998); Iosevich and Sawyer (1996); Tao and Wright (2003); Dendrinos, Laghi and Wright (2009); R. Oberlin and Erdogan (2010); Stovall (2010, 2011)

L^p-Improving: Present Landscape

- The case of averages over curves is now well-understood (with the exception of endpoints in some cases).
- Averages over nondegenerate families of hypersurfaces are also well-understood. Many cases of degenerate families of hypersurfaces have been studied, but no complete picture has emerged.
- Aside from curves and hypersurfaces, work is sparse and frequently tied to specific examples with very nice properties. This is in part due to combinatorial limitations of refinement/expansion methods. It is also in part due to the fact that the associated FIOs are generally more degenerate than expected even in very low codimension.

Goals

- Make inroads on the problem of intermediate dimension. This is difficult because there are occasionally strange things that happen here and there's no obvious way to quantify exactly what "well-curved" means here.
- Move away from the usual limitations of refinements (i.e., avoid having to make explicit algebraic constraints on dimension and codimension; alternately, find a way to refine/inflate that doesn't change dramatically as dimension and codimension change).
- As much as possible, establish results which are stable under suitably small perturbations of the geometry.

Hyperinflation and TT^*T

- **For convolution with the standard measure on the submanifold** $(t_1, t_2) \mapsto (t_1, t_2, t_1^2, t_2^2, t_1 t_2)$**, naive inflation doesn't work because** 5 **is odd.** No number of copies of the two-dimensional map can combine to give a nondegenerate map into $\mathbb{R}^5$. There are more sophisticated ways to inflate, but none of these seem to be productive either.
- **For** $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1^2, t_2^2, t_3^2, t_1 t_2, t_2 t_3, t_1 t_3)$**, inflation doesn't work because the Jacobian vanishes identically.** Adding three copies of this map, though the dimensions are favorable, doesn't work because the Jacobian is identically zero. Three copies of this map don't actually fill $\mathbb{R}^9$.
- **It turns out that results are possible by over-inflating.**
- **The operator** TT^*T **is superior to** TT^* **in this case because it is more similar to** T **geometrically.**

Refinements Refined I

The general approach is a TT^*T version of refinements:

Lemma (Generalized TT^*T)

Suppose T is a positive linear operator which maps $L^2(X_L)$ to $L^2(X_R)$. For any measurable sets F and G in X_L and X_R with finite, nonzero measure, let

$$F' := \left\{ x \in F \;\middle|\; T^* \chi_G(x) \geq \frac{\int_G T\chi_F}{3|F|} \right\} \text{ and}$$

$$G' := \left\{ y \in G \;\middle|\; T \chi_F(y) \geq \frac{\int_G T\chi_F}{3|G|} \right\}.$$

Then

$$\left(\frac{1}{3} \int_G T\chi_F \right)^3 \leq |F||G| \int_G T(\chi_{F'} T^*(\chi_{G'} T\chi_F)).$$

This lemma stops just short of making refinements a black box.

Refinements Refined II

Proof

Let
$$\delta_F := \frac{1}{3|F|} \int_G T\chi_F \text{ and } \delta_G := \frac{1}{3|G|} \int_G T\chi_F.$$
It follows that
$$\begin{aligned}\int_G T_{GF} T^*_{G'F'} T_{GF}\chi_F &= \int_{F'} (T^*\chi_G)(T^*_{G'F'} T_{GF}\chi_F) \\ &\geq \delta_F \int_{F'} (T^*_{G'F'} T_{GF}\chi_F) \\ &= \delta_F \int_{G'} (T_{G'F'}\chi_{F'})(T\chi_F) \geq \delta_F \delta_G \int_{G'} T\chi_{F'}\end{aligned}$$
and
$$\begin{aligned}\int_{G'} T\chi_{F'} &= \int_G T\chi_F - \int_{G\setminus G'} T\chi_F - \int_{F\setminus F'} T^*\chi_{G'} \\ &\geq \int_G T\chi_F - \delta_G|G| - \delta_F|F| \geq \frac{1}{3}\int_G T\chi_F.\end{aligned}$$

Iterated Incidence Manifold

The operator TT^*T is naturally connected to a submanifold $\mathcal{M}^{(3)}$ of $X_R \times X_L \times X_R \times X_L$:

$$\mathcal{M}^{(3)} := \left\{ (y^{(2)}, x^{(2)}, y^{(1)}, x^{(1)}) \;\middle|\; x^{(2)} \in \Sigma_{y^{(2)}}; x^{(1)}, x^{(2)} \in \Sigma_{y^{(1)}} \right\}.$$

In this notation, the quantity that must be estimated is

$$\int_{\mathcal{M}^{(3)}} \chi_G(y^{(2)}) \chi_{F'}(x^{(2)}) \chi_{G'}(y^{(1)}) \chi_F(x^{(1)}) d\mu$$

where μ is some measure of smooth density on $\mathcal{M}^{(3)}$.

Main Dimensional Constraint

The proof needs $\dim \mathcal{M}^{(3)} \geq \dim X_L + \dim X_R$, which corresponds to averages over submanifolds of dimension at least $1/3$ the ambient dimension.

Application of Coarea Formula

Assuming the dimensional constraint, we Fubinate inside the TT^*T object to integrate over the inner variables first. In other words, write

$$\int_{\mathcal{M}^{(3)}} \chi_G(y^{(2)})\chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\chi_F(x^{(1)}) d\mu$$
$$= \int_{X_R \times X_L} \chi_G(y)\chi_F(x) B_{y,x}(\chi_{F'}, \chi_{G'}) d\mu_R(y) d\mu_L(x),$$

where

$$B_{y,x}(\chi_{F'}, \chi_{G'}) := \int_{x, x^{(2)} \in \Sigma_{y^{(1)}}; x^{(2)} \in \Sigma_y} \chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)}) \frac{d\mathcal{H}^k}{J}$$

comes from the coarea formula. Unlike the previous integrals, the geometric structure and associated measure inside B are both **very bad**, especially when $x \in \Sigma_y$.

Example: Convolution with Nondegenerate 2-surface in $\mathbb{R}^5$

- Consider the convolution operator given by

$$\gamma(t_1, t_2) := (t_1, t_2, t_1^2, t_1 t_2, t_2^2) \text{ and } Tf(y) := \int_{\mathbb{R}^2} f(y + \gamma(t))dt.$$

- Fix any $x = (x_1, x_2, x_{11}, x_{12}, x_{22})$ in $\mathbb{R}^5$; similarly for y. Define

$$\delta_{ij} := x_{ij} - y_{ij} - (x_i - y_i)(x_j - y_j), \qquad M := \begin{bmatrix} \delta_{22} & -\delta_{12} \\ -\delta_{12} & \delta_{11} \end{bmatrix}.$$

- Solve

$$\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} M \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} 0 & \det M \\ \det M & 0 \end{bmatrix}.$$

- Up to symmetry in u and v,

$$B_{y,x}(\chi_{F'}, \chi_{G'}) = \int \chi_{F'}(y + \gamma(su + x - y))\chi_{G'}(x - \gamma(s^{-1}v + x - y))\frac{ds}{s}.$$

The Road So Far

Recall

$$\frac{1}{27|F||G|}\int_G T\chi_F \leq \int_{\mathcal{M}^{(3)}} \chi_G(y^{(2)})\chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\chi_F(x^{(1)})d\mu$$
$$\leq \int_{X_R\times X_L} \chi_G(y)\chi_F(x)\left[B_{y,x}(\chi_{F'},\chi_{G'})\right] d\mu_R(y)d\mu_L(x)$$

where

$$B_{y,x}(\chi_{F'},\chi_{G'}) := \int_{x,x^{(2)}\in\Sigma_{y^{(1)}};x^{(2)}\in\Sigma_y} \chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\frac{d\mathcal{H}^k}{J}$$

Major Obstacles Ahead:

- One can say essentially nothing about the submanifolds over which the integral in B is taken.
- Rough estimation of J is possible, but can say essentially nothing about how it varies inside the integral.

General Fiber Integrals I

- Assuming that S is some general k-dimensional submanifold of Euclidean space (e.g., no control on geometric quantities like curvature and possibly even topological quantities like number of connected components), what can we say about k-dimensional Hausdorff measure on S?
- This is the analogous problem to counting solutions of systems of equations (for which Bézout's Theorem or similar tools are frequently used).
- It turns out that the right thing to study is the regularity of $\mathcal{H}^k$, i.e., to establish that

$$\mathcal{H}^k(S \cap B_r(x)) \lesssim r^k \qquad \forall (x,r) \in \mathbb{R}^n \times (0,\infty).$$

Of course, the bad news is that this inequality is **not necessarily true**. What we want is a qualitative condition on S which guarantees this quantitative result.

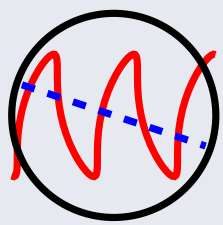
General Fiber Integrals II

Regularity of the measure on the fibers happens exactly when a Bézout-type finiteness condition on systems of equations can be shown to hold:

Lemma

If S is any k-dimensional immersed submanifold in $\mathbb{R}^d$ (not necessarily connected or compact) such that S transversely intersects any affine $(d-k)$-dimensional subspace at most m times, then

$$\mathcal{H}^k(S \cap B_r(x)) \leq C_{k,d} m r^k.$$



$\mathcal{H}^1(S \cap B_r(x)) \leq 5\pi r$

In other words, wadding / winding / folding are the only ways to fit a long rope or a large map in a small box.

General Fiber Integrals III

Proof Part I

Suppose $\gamma : U \to \mathbb{R}^d$ parametrizes a piece of S. Then

$$\begin{aligned}
\mathcal{H}^k(\gamma(S)) &= \int_U \left|\left|\frac{\partial \gamma}{\partial t}\right|\right| dt \\
&= \int_U \sup_{\substack{||\omega_i|| \leq 1 \\ i=1,\ldots,d-k}} \left| \det \left[\frac{\partial \gamma}{\partial t_1}, \ldots, \frac{\partial \gamma}{\partial t_k}, \omega_1, \ldots, \omega_{d-k} \right] \right| dt \\
&= C_{d,k} \int_{O(d)} \left[\int_U \left| \det \left[\frac{\partial \gamma}{\partial t_1}, \ldots, \frac{\partial \gamma}{\partial t_k}, \omega_1, \ldots, \omega_{d-k} \right] \right| dt \right] d\sigma(\omega) \\
&= C_{d,k} \int_{O(d)} \left[\int_U \left| \det \frac{\partial P_\omega \gamma}{\partial t} \right| dt \right] d\sigma(\omega)
\end{aligned}$$

where P_ω is orthogonal projection onto the final k columns of $\omega \in O(d)$.

General Fiber Integrals IV

Proof Part II

$$\mathcal{H}^k(\gamma(S)) = C_{d,k} \int_{O(d)} \left[\int_U \left| \det \frac{\partial P_\omega \gamma}{\partial t} \right| dt \right] d\sigma(\omega)$$

Using a partition of unity, the change of variables formula and summing,

$$\mathcal{H}^k(S) = C_{d,k} \int_{O(d)} \left[\int_{P_\omega(S)} N_\omega(x) dx \right] d\sigma(\omega),$$

where $N_\omega(x)$ is the number of transverse intersections of S with the affine subspace

$$\left\{ y \in \mathbb{R}^d \;\mid\; \langle \omega_{d-k+j}, y \rangle = x_j, \; j = 1, \ldots, k \right\}$$

If $N_\omega(x) \leq m$ and $S \subset B_r(x)$, then $\mathcal{H}^k(S) \leq C_{d,k} m r^k$ as desired.

Back to the Main Estimates

Recall

$$\frac{1}{27|F||G|}\int_G T\chi_F \leq \int_{\mathcal{M}^{(3)}} \chi_G(y^{(2)})\chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\chi_F(x^{(1)})d\mu$$
$$\leq \int_{X_R\times X_L} \chi_G(y)\chi_F(x)\left[B_{y,x}(\chi_{F'},\chi_{G'})\right] d\mu_R(y)d\mu_L(x)$$

where

$$B_{y,x}(\chi_{F'},\chi_{G'}) := \int_{x,x^{(2)}\in\Sigma_{y^{(1)}};x^{(2)}\in\Sigma_y} \chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\frac{d\mathcal{H}^k}{J}$$

What we can say: If D is distance from (y,x) to $(y^{(1)},x^{(2)})$, then

$$\int_{x,x^{(2)}\in\Sigma_{y^{(1)}};x^{(2)}\in\Sigma_y} \chi_{F'}(x^{(2)})\chi_{G'}(y^{(1)})\frac{d\mathcal{H}^k}{D^k} \lesssim 1$$

uniformly in (y,x). [This is a small lie which ignores logarithmic growth as (y,x) approaches the set where $x\in\Sigma_y$.]

Reduction to Sublevel Set Estimates I

If we let $\mathcal{M}^{(3)}_\alpha$ be the (bad) subset of $\mathcal{M}^{(3)}$ on which $J \leq \alpha D^k$,

$$\int_{\mathcal{M}^{(3)} \setminus \mathcal{M}^{(3)}_\alpha} \chi_G(y^{(2)}) \chi_{F'}(x^{(2)}) \chi_{G'}(y^{(1)}) \chi_F(x^{(1)}) d\mu \lesssim \alpha^{-1} |G||F|.$$

For the integral on the bad set $\mathcal{M}^{(3)}_\alpha$, we regard $(x^{(2)}, y^{(1)})$ as fixed and think of the object as a sublevel set operator:

$$W_{x^{(2)}, y^{(1)}}(\chi_G, \chi_F) :=$$
$$\int_{y^{(2)} \in \Sigma^*_{x^{(2)}}} \int_{x^{(1)} \in \Sigma_{y^{(1)}}} \chi_{\frac{J}{D^k} \leq \alpha} \chi_G(y^{(2)}) \chi_F(x^{(1)}) d\sigma^*(y^{(2)}) d\sigma(x^{(1)}).$$

Notice also that

$$\int_{y^{(2)} \in \Sigma^*_{x^{(2)}}} \chi_G(y^{(2)}) d\sigma(y^{(2)}) = T^* \chi_G(x^{(2)})$$

and likewise the other integral is $T \chi_F(y^{(1)})$.

Reduction to Sublevel Set Estimates II

Assuming that the sublevel set operators are bounded, the sort of inequality we get is:

$$\int_G T_{GF} T^*_{G'F'} T_{GF} \chi_F \lesssim \frac{1}{\alpha}(|F||G|)^{1-\epsilon} + \alpha^s \int |T^* \chi_G|^{\frac{1}{p_l}} T^*_{G'F'} |T \chi_F|^{\frac{1}{p_r}}$$

Now, **because of the G' and F'** we may replace

$$|T^* \chi_G|^{\frac{1}{p_l}} \rightsquigarrow \delta_F^{-\frac{1}{p_l'}} T^* \chi_G \text{ and } |T \chi_F|^{\frac{1}{p_r}} \rightsquigarrow \delta_G^{-\frac{1}{p_r'}} T \chi_F$$

which essentially allows for a bootstrapping-type inequality if we choose α so that

$$\alpha^s \delta_F^{-\frac{1}{p_l'}} \delta_G^{-\frac{1}{p_r'}} \ll 1.$$

By the TT^*T refinement inequality, this gives an upper bound for $\int_G T \chi_F$. Because δ_F, δ_G both contain $\int_G T \chi_F$, it's another bootstrapping-type situation (but we know the quantity must be finite). This leads to a restricted weak type estimate for T.

Understanding the Jacobian

$$W_{x^{(2)},y^{(1)}}(\chi_G,\chi_F) :=$$
$$\int_{y^{(2)}\in\Sigma^*_{x^{(2)}}}\int_{x^{(1)}\in\Sigma_{y^{(1)}}}\chi_{\frac{J}{D^k}\leq\alpha}\chi_G(y^{(2)})\chi_F(x^{(1)})d\sigma^*(y^{(2)})d\sigma(x^{(1)}).$$

- X_L^i: vector fields tangent to $\mathcal{M} := \{(y,x) \mid x \in \Sigma_y\}$ which project to zero in the space X_L (second factor).
- X_R^i: vector fields tangent to $\mathcal{M}$ which project to zero in the space X_R (first factor).
- Roughly speaking, W reduces to the object

$$W(\chi_G,\chi_F) \approx \tilde{W}(\chi_{\tilde{G}},\chi_{\tilde{F}}) := \int \chi_{\tilde{G}}(t)\chi_{\tilde{F}}(s)\chi_{\Phi\leq\alpha}dtds$$

where

$$\Phi \approx \frac{1}{(|t|+|s|)^k}\mathrm{vol}_{T_p(\mathcal{M})}\left\{X_L^*,X_R^*,\left[\sum_i t_iX_L^i,X_R^*\right],\left[X_L^*,\sum_i s_iX_R^i\right]\right\}$$

to lowest order in t and s.

Convolution with Nondegenerate 2-surface in $\mathbb{R}^5$ II

Recall

$$W(\chi_G, \chi_F) \approx \tilde{W}(\chi_{\tilde{G}}, \chi_{\tilde{F}}) := \int \chi_{\tilde{G}}(t)\chi_{\tilde{F}}(s)\chi_{\Phi \leq \alpha} dtds$$

$$\Phi \approx \frac{1}{(|t| + |s|)^k} \mathrm{vol}_{T_p(\mathcal{M})} \left\{ X_L^*, X_R^*, \left[\sum_i t_i X_L^i, X_R^* \right], \left[X_L^*, \sum_i s_i X_R^i \right] \right\}$$

For the two surface in $\mathbb{R}^5$, there are two each of X_L^i and X_R^i, $k = 1$, and $T_p(\mathcal{M})$ is seven dimensional. If we take

$$X_L^1, X_L^2, X_R^1, X_R^2, \left[\sum_i t_i X_L^i, X_R^1 \right], \left[\sum_i t_i X_L^i, X_R^2 \right]$$

to build a spanning set, may as well take

$$\left[\frac{-t_2 X_L^1 + t_1 X_L^2}{|t|}, \sum_i s_i X_R^i \right]$$

as the final vector.

Convolution with Nondegenerate 2-surface in $\mathbb{R}^5$ III

Unfortunately it is not possible to devise a constraint on the commutators $[X_L^i, X_R^j]$ so that

$$X_L^*, X_R^*, \left[\sum_i t_i X_L^i, X_R^*\right], \left[\frac{-t_2 X_L^1 + t_1 X_L^2}{|t|}, \sum_i s_i X_R^i\right]$$

always span for any pair (t, s). However, for any t, you can insist that it is always possible to find **some** s, which is effectively the same as saying

$$|\nabla_s \Phi(t,s)| \gtrsim |t|.$$

You can also insist that $|\nabla_t \Phi(t,s)| \gtrsim |s|$, which implies an estimate

$$\int \chi_{\tilde{G}}(t) \chi_{\tilde{F}}(s) \chi_{\Phi \leq \alpha} dtds \leq \alpha |\tilde{G}|^{\frac{1}{2}} |\tilde{F}|^{\frac{1}{2}}.$$

This estimate gives an $L^{\frac{8}{5}} \to L^{\frac{8}{3}}$ estimate up to infinitesimal loss, which is best possible.

Concluding Remarks

- The technique sketched here works in a number of other cases (see arXiv:1609.02972). However many related problems are far from resolved, even some multilinear, one-dimensional problems.
- The only limitation in proving rather general results (i.e., in terms of knowledge of v.f. commutators only) is whether the limited information it gives about Φ is stable enough to prove a sublevel set functional inequality.
- Ultimately one would like to know if there is a unified way of identifying what estimates a given operator satisfies (a la Tao and Wright (2003), for example). ~~There seems to be mounting evidence that this may not be possible to do in a very general way (or is, at the moment, beyond reach).~~