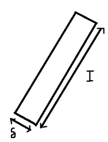
An improved bound on the Hausdorff dimension of Besicovitch sets in $\mathbb{R}^3$: Joshua Zahl, 19 May 2017

joint work with Nets Katz

A Besicovitch set in $\mathbb{R}^n$ is a compact set containing a line segment pointing in every direction.

Conjecture. *Every Besicovitch set in $\mathbb{R}^n$ must have Hausdorff dimension n.*

Theorem. *(Katz, Zahl) Every Kakeya set in $\mathbb{R}^3$ has H-dimension$\geq \frac{5}{2} + \epsilon_0$, where $\epsilon_0 > 0$.*



Let $\delta > 0$. A tube is a δ-neighborhood of a unit line segment.

Definition. *A set Π of tubes satisfies the Wolff axioms if (1) every tube in Π is in $B(0,1)$, (2) for every rectangular prism R of dimensions $(2, s, t)$, at most $\delta^{-2}st$ tubes from Π are contained in R.*

Theorem. *(Katz, Zahl) Let Π be a set of tubes that satisfy the Wolff axiom. If $|\Pi| = \delta^{-2}$, then $\delta^{\frac{1}{2}-\epsilon_0} \lesssim |\cup_{T\in\Pi} T|$.*

For each of the tubes, let $Y(T)T$. If $\sum_{T\in\Pi} |Y(T)| > \lambda$ then

$$\lambda^c \delta^{\frac{1}{2}-\epsilon_0} \lesssim |\cup_{T\in\Pi} Y(T)| \tag{1}$$

Theorem. *(Katz, Laba, Tao) Let Π be a set of δ^{-2} tubes pointing in δ-separated directions. Then,*

$$|\cup_{T\in\Pi} T| \geq \delta^{\frac{1}{2}-\epsilon_1} \text{ or}$$
$$|\cup_{T\in\Pi} N_{\delta^{\frac{1}{2}}}(T)| \geq (\delta^{\frac{1}{2}})^{\frac{1}{2}-\epsilon_1}.$$

Heisenberg Group: $\mathbb{H} = \{(x, y, z) \in \mathbb{C}^3, Im(z) = Im(x\overline{y})\}$
If $a, b \in \mathbb{R}$, $c \in \mathbb{C}$, the line $(0, w, b) + \mathbb{C}(1, a, \overline{w}) \subset \mathbb{H}$.

Let $R = \mathbb{F}_p[t]/(t^2)$. If $a \in R$,$a = a_1 + a_2t$ where $a_1, a_2 \in \mathbb{F}_p$.

$X = \{(x_1 + x_2t, y_1 + y_2t, z_1 + z_2t)|z_2 = x_1y_2 - y_1x_2\}$.
If $a, b, c, d \in \mathbb{F}_p$, $ad - bc = 1$, then $(a + \alpha at, b + \alpha bt) + R(c + \alpha ct, d + \alpha dt)X$.
$\Pi : \mathbb{R}^3 \to \mathbb{F}_p^3$, $\Pi(x) = \mathbb{F}_p^3$.

Vague Theorem. *If Π is a counter example to the above theorem, then it is either the Heisenberg or SL_2 example.*

A regulus is a (quadric) surface in $\mathbb{R}^3$ that is doubly ruled by lines.

If L_1, L_2, L_3 are skew lines then the union of the lines incident to L_1, L_2, L_3 form a regulus.

$H(T_0)$ is the "'hairbrush of T_0", the set of tubes from Π that hit T_0.

A regulus strip is a set of the form $N_\delta(Z) \cap N_{\delta^{\frac{1}{2}}}(L) \cap B(0,1)$, where $N_\delta(Z)$ is a regulus and $N_{\delta^{\frac{1}{2}}}(L)$ is a line in Z.

If T is a SL_2 type set Π is a disjoint union of $\delta^{-\frac{1}{2}}$ sets each of which is contained in a regulus strip.

Lines in $\mathbb{R}^3 \leftrightarrow$ points in $\mathbb{R}^4$.

Tubes in $\mathbb{R}^3 \leftrightarrow$ δ-balls in $\mathbb{R}^4$

If T is a SL_2 type counter example then $image(T)$ in $\mathbb{R}^4$ is contained in $N_{\delta^{\frac{1}{2}}}(Z(p))$. $P(a,b,c,d) = ad - bc - 1$.