Notes for talk "Flag Area Measures" by Susanna Dann (joint w/ J. Abardia, A. Bernig)

$$S_k^{(p)}(P,\cdot) \quad \sim \quad \cos^2(\varepsilon_1, \varepsilon_2) \quad \text{Hilbert's flag area measures}$$

$$S_k^{(p);i}(P,\cdot) \quad \sim \quad \sigma_i(\cos^2\theta_1, \dots, \cos^2\theta_{n-1})$$

principal angles between subspaces

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- We consider $(\mathbb{R}^n, SO(n))$, $\mathcal{K}^n$ ← space of convex bodies in $\mathbb{R}^n$

Def: $\mu : \mathcal{K}^n \to \mathbb{R}$ is a valuation if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

holds for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.

Properties of valuations: Let $K \in \mathcal{K}^n$, let $\mu : \mathcal{K}^n \to \mathbb{R}$ be a valuation.

(i) $\mu(K+y) = \mu(K)$, $y \in \mathbb{R}^n$

(ii) $\mu(gK) = \mu(K)$, $g \in SO(n)$

- $\mathrm{Val}^{SO(n)} = \mathrm{span}\{\mu_0, \dots, \mu_n\}$

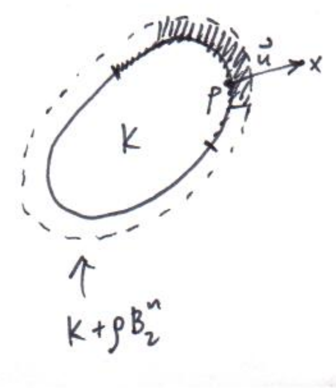
- $\mathrm{vol}(K + \rho B_2^n) = \sum_{k=0}^{n} c_{n,k} \overbrace{\mu_k(K)}^{k\text{-hom.}} \rho^{n-k}$

- Let $\eta \subset \mathbb{R}^n \times S^{n-1}$. Put $M_\rho(K,\eta) := \{x \in \mathbb{R}^n : 0 < d(K,x) \leq \rho,\ (p,u) \in \eta\}$.

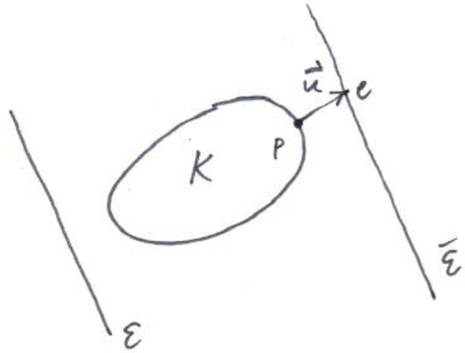
Then $\mathrm{vol}(M_\rho(K,\eta)) = \sum_{k=0}^{n-1} c_{n,k} \underbrace{\Theta_k(K,\eta)}_{\text{support measures}} \rho^{n-k}$

- $\beta \subset S^{n-1}$, $S_k(K,\beta) := \Theta_k(K, \mathbb{R}^n \times \beta)$

$K \mapsto S_k(K,\cdot)$



• Gr_p^n , $\overline{Gr_p^n}$



$\eta \subset \mathbb{R}^n \times S^{n-1} \times Gr_p^n$

$$\mathcal{M}_\rho^{(p)}(K,\eta) = \{ \bar{\varepsilon} \in Gr_p^n : 0 < d(K,\bar{\varepsilon}) \le \rho, \; (p,u,\varepsilon) \in \eta \}$$

• $$\mathrm{vol}\left(\mathcal{M}_\rho^{(p)}(K,\eta)\right) = \sum_{k=0}^{n-p-1} c_{n,p,k} \underbrace{\Xi_k^{(p)}(K,\eta)}_{\text{flag (type) support measure}} \rho^{n-p-k}$$

• $\beta \subset S^{n-1} \times Gr_p^n$, $S_k^{(p)}(K,\beta) := \Xi_k^{(p)}(K, \mathbb{R}^n \times \beta)$

flag type area measures

$$K \mapsto S_k^{(p)}(K,\cdot)$$

• flag manifold $\mathcal{F}_{1,p+1} := \{ (v,\varepsilon) \in S^{n-1} \times Gr_{p+1}^n : v \in \varepsilon \}$

$$\mathcal{F}_{1,p}^{\perp} := \{ (v,\varepsilon) \in S^{n-1} \times Gr_p^n : v \perp \varepsilon \}$$

Define a mapping $\mathcal{F}_{1,p}^{\perp} \longrightarrow \mathcal{F}_{1,p+1}$ by

$$(v,\varepsilon) \longmapsto (v, \mathrm{span}\{v,\varepsilon\})$$

<u>Theorem</u> (Hinderer–Hug–Weil): Let $0 \le p \le n-1$, $0 \le k \le n-p-1$, P polytope, $\beta \in \mathcal{B}(\mathcal{F}_{1,p}^{\perp})$.

Then $$S_k^{(p)}(P,\beta) = c_{n,p,k} \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}(F) \int_{n(P,F)} \int_{Gr_{p+1}(v)} \mathbb{1}_{(v,\varepsilon \cap v^\perp) \in \beta} \cos^2(\varepsilon^\perp, F) \, d\varepsilon \, dv.$$

Principal angles

$\mathcal{E} \in Gr_p^n$, $e_1, \dots, e_p$ orthonormal basis (ONB)
$F \in Gr_k^n$, $f_1, \dots, f_k$ ONB

$\Big\}\ i \neq j : \langle e_i, f_j \rangle = 0$

$\langle e_i, f_i \rangle = \cos\theta_i \,, \quad 1 \le i \le M := \min\{k, n-k, p, n-p\}$

$\theta_1, \dots, \theta_M$ principal angles between $\mathcal{E}$ and F

$\sigma_i(\mathcal{E}, F) = \sigma_i(\cos^2\theta_1, \dots, \cos^2\theta_M)$

Def: $\mathcal{P} : \mathcal{K}^n \to \mathrm{Meas}(\mathcal{F}_{1,p+1})$ is a flag area measure if it is a continuous translation-invariant valuation, Flag Area$^{(p)}$.

Thm 1: $\forall\, 0 \le k, p \le n-1$, $0 \le i \le M := \min\{k, n-1-k, p, n-1-p\}$, $\exists!$ flag area measure s.t. for any polytope P, $\beta \subset \mathcal{B}(\mathcal{F}_{1,p+1})$,

$$S_k^{(p),i}(P, \beta) = c_{n,p,k,i} \sum_{F \in \mathcal{F}_k(P)} \mathrm{vol}_k(P) \int_{n(P,F)} \int_{Gr_{p+1}(v)} \mathbb{1}_{(v,\mathcal{E}) \in \beta}\, \sigma_i(\cos^2\theta_1, \dots, \cos^2\theta_M)\, d\mathcal{E}\, dv.$$

- If n is odd, $p = k = \frac{n-1}{2}$, then $\widetilde{S}(P, \beta) = $ — " — $\widetilde{\sigma}(\mathcal{E}^\perp, F) \dots$

Thm 2: (i) $S_k^{(p),M}(K, \eta) = S_k^{(p)}(K, \eta)$

(ii) $S_k^{(p),i}(\cdot, \beta)$, $\widetilde{S}(\cdot, \beta)$ are cont. trans.-inv. valuation, hom. of k, $\frac{n-1}{2}$

(iii) $S_k^{(p),i}(gK, g\beta) = S_k^{(p),i}(K, \beta)$, $g \in O(n)$

(iv) $\widetilde{S}(gK, g\beta) = \det(g)\, \widetilde{S}(K, \beta)$, $g \in O(n)$

(v) Let $\pi : \mathcal{F}_{1,p+1} \to S^{n-1}$, $(v, \mathcal{E}) \mapsto v$, $\beta \subset \mathcal{B}(S^{n-1})$. Then

$$S_k^{(p),i}(K, \pi^{-1}(\beta)) = S_k(K, \beta) \text{ and } \widetilde{S}(K, \pi^{-1}(\beta)) = 0.$$

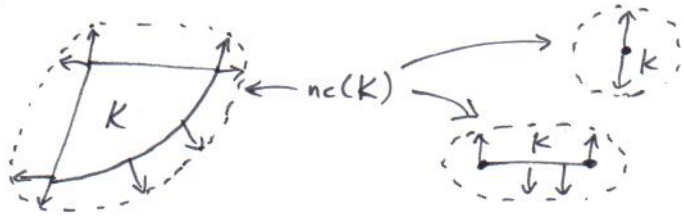
<u>Def</u>: $\mathcal{P} \in \mathrm{Flag\,Area}^{(p)}$ is <u>smooth</u> if $\exists$ trans.-inv. form $\tau \in \Omega^*(\mathbb{R}^n \times \mathcal{F}_{1,p+1})$ s.t.

$$\int_{\mathcal{F}_{1,p+1}} f \, d\mathcal{P}(K,\cdot) = \int_{\underset{\substack{\uparrow \\ \text{normal cycle of } K}}{nc(K)}} \pi_*(f \wedge \tau), \quad \forall K \in \mathcal{K}^n, \ f \in C^\infty(\mathcal{F}_{1,p+1}).$$

<u>Thm 3</u>: If $(p,k) \neq (\frac{n-1}{2}, \frac{n-1}{2})$ then $S_k^{(p),i}$ forms a basis for $\mathrm{Flag\,Area}_k^{(p),SO(n),\mathrm{smooth}}$.

If n is odd, $p = k = \frac{n-1}{2}$, then $S_{\frac{n-1}{2}}^{(\frac{n-1}{2}),i}$ and $\tilde{S}$ forms a basis for $\mathrm{Flag\,Area}_k^{(p),SO(n),\mathrm{smooth}}$.

$K \subset \mathbb{R}^n$, $nc(K) \subset \mathbb{R}^n \times S^{n-1}$, $\dim(nc(K)) = n-1$



Ex) $\omega \in \Omega^{n-1}(\mathbb{R}^n \times S^{n-1})$, $K \mapsto \int_{nc(K)} \omega \in \mathbb{R}$ smooth valuation

$\eta \subset S^{n-1}$, $\int_{nc(K) \cap (\mathbb{R}^n \times S^{n-1})} \omega$ smooth area measure

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$$\begin{array}{ccc} & \mathbb{R}^n \times SO(n) & \sigma_i, \ 1 \le i \le n \quad , \quad \omega_{ij}, \ 1 \le i < j \le n \\ (x, v, \varepsilon) & \downarrow & \\ \downarrow & \mathbb{R}^n \times \mathcal{F}_{1,p+1} & \\ (x,v) & \downarrow \pi, \ r = \dim(\text{fiber}) & \\ & \mathbb{R}^n \times S^{n-1} & \end{array}$$

$\pi^*\tau \in \Omega^k(\mathbb{R}^n \times \mathcal{F}_{1,p+1})$

$\uparrow$

$\tau \in \Omega^k(\mathbb{R}^n \times S^{n-1})$

$\tau \in \Omega^{k+r}(\mathbb{R}^n \times \mathcal{F}_{1,p+1})$

$\vdots$

$\pi_* \tau \in \Omega^k(\mathbb{R}^n \times S^{n-1})$