Will Sawin: Applications of Exponential Sums

ETH-Zürich

Klosterman sums:

$$\mathrm{Kl}_K(a;q) := \frac{1}{q^{\frac{K-1}{2}}} \sum_{\substack{x_1,\dots,x_K \\ \prod_{i=1}^K x_i = a}} \psi\left(\sum_{i=1}^K x_i\right), \qquad \psi : \mathbb{F}_q \to \mathbb{C}^\times \text{ character}$$

Thm 1 (Kowalski-Michel-S.)

q prime, α_n, β_m fns.

$$\left| \sum_{m \le M} \sum_{n \le N} \alpha_n \beta_m \mathrm{Kl}_K(mn;q) \right| \le \|\alpha\|_2 \|\beta\|_1 O\left(\sqrt{MN}\, q^{\varepsilon} \times \left(M^{-1/2} + (MN)^{-3/16} q^{11/64}\right)\right)$$

Bound is nontrivial if $MN \ge q^{11/12}$

Applications.

Thm 2 (Blomer et al, ~~Kowalski~~ K-M-S)

f a cusp form (holom. or Maass) level 1, q prime

$$\frac{1}{q-1} \sum_{\chi \,(\mathrm{mod}\, q)} \left| L(f \otimes \chi, 1/2) \right|^2 = \frac{2 L(\mathrm{Sym}^2 f; 1)}{\zeta(2)} \log q + \beta_f + O_f\left(q^{-1/145}\right)$$

Expected main term via residues

explicit constant

If g is another such form, $(f \neq g)$

$$\frac{1}{q-1} \sum_{\chi (q)} L(f \otimes \chi, \tfrac{1}{2}) \overline{L(g \otimes \chi, \tfrac{1}{2})} = \frac{2 L(f \otimes g ; 1)}{\zeta(2)} + O(q^{-1/145})$$

unless sum is trivially 0,
i.e. due to f, g both holom w/ different chars.

<u>Thm 3</u> (KMS)

f cusp form eigenform level 1, λ_f its Fourier coeffs., normalized s.t. $\lambda_f(n) = n^{o(1)}$

$$\lambda_f * 1(n) = \sum_{d \mid n} \lambda_f(d),$$

q prime, $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, $X, A \in \mathbb{R}$.

$$\sum_{\substack{n \leq X \\ n \equiv a \ (\mathrm{mod}\ q)}} \lambda_f * 1(n) = \frac{1}{q-1} \Bigg(\sum_{\substack{n < X \\ (n,q)=1}} \lambda_f * 1(n) \Bigg) + O_{A,f}\left(\tfrac{X}{q} \log X^{-A}\right)$$

for $q < X^{1/2 + 1/103}$

<u>Sketch of Thm 1 $\Rightarrow$ Thm 3</u>

1) Expand sum on left

$$\sum_{n,m} \lambda_f(n) \mathbb{1}_{nm \leq X} \mathbb{1}_{nm \equiv a \ (\mathrm{mod}\ q)}$$

2) Apply Voronoi summation formula (good analog of Poisson summation for mod. forms)

3) This causes $Kl_3(x;q)$ and Bessel fns. to appear
since we have a form on GL_3.

4) Other terms become α_n, β_m in Thm 1 $\rightarrow$ main term

Rougher Sketch of Thm 1 $\Rightarrow$ Thm 2

- Use approx. func'l eqn. to reduce $L(f \otimes \chi, 1/2)$ to a short sum.
- Split into various ranges, one of which fits shape of Thm 1 $\rightarrow$ main term
- Other methods needed for remaining ranges

Proof ~~Sketch~~ Sketch of Thm 1

Reduce to complete exponential sums. Note there is some symmetry in m, n.

$$\sum_{m \le M} \sum_{n \le N} \alpha_n \beta_m Kl_k(mn; q)$$

By Cauchy-Schwarz, sufficient to bound

$$\sum_{n_1, n_2 \le N} \alpha_{n_1} \overline{\alpha_{n_2}} \left(\sum_{m \le M} Kl_k(mn_1) Kl_k(mn_2) \right)$$

$$\approx \frac{1}{AB} \sum_{A<a<2A} \sum_{B<b<2B} \sum_{n_1, n_2} \alpha_{n_1} \overline{\alpha_{n_2}} \sum_{m<M} Kl_k((m+a)n_1) Kl_k((m+b)n_2)$$

with $AB \ll M$

Change of variables $m = ar$, $n_1 = a^{-1}s_1$, $n_2 = a^{-1}s_2$

$r = a^{-1}m$, $s_1 = an_1$, $s_2 = an_2$

Extend to $s_i \le AN$

gives $\cancel{\sum_{B<b<2B} Kl_k}$

$$\underbrace{\sum_{r, s_1, s_2} \left(\quad \right)}_{\text{Roughly uniform/ easy}} \underbrace{\sum_{B<b<2B} Kl_k((r+b)s_1) Kl_k((r+b)s_2)}_{\text{Use Hölder, ..}}$$

In particular, take $\left(\sum_b Kl_k((r+b)s_1) \overline{Kl_k((r+b)s_2)} \right)^4$,

expand / compare to 8-fold sum!

<u>Lemma 4</u> $\vec{b} = (b_1, b_2, b_3, b_4)$

Any character (Not 4 from before)

$$R(r, \lambda, \vec{b}) := \sum_{s \in \mathbb{F}_q} Kl_k(s(r+b_1)) Kl_k(s(r+b_2)) \overline{Kl_k(s(r+b_3)) Kl_k(s(r+b_4))} \psi(\lambda s)$$

$$\sum_{r \in \mathbb{F}_q} R(r, \lambda_1, \vec{b}) \overline{R(r, \lambda_2, \vec{b})} = q^2 \delta_{\lambda_1 \lambda_2} + O(q^{3/2})$$

for $\vec{b}$ outside of some "bad" algebraic subset

<u>Note</u>: $|R| \sim \sqrt{q}$

- Assuming stronger forms of Lemma 4,
  Thm 1 should have cancellation for $MN \geq q^{3/4}$
  (e.g. $M = N = q^{3/8}$)

General cancellation principle: (algebraic geom.)
n-variable product, k "symmetries"
Expected bound: $q^{n/2 - k/4}$. Thm 1 is $n=2$, $k=1$

<u>Proof Ideas for Lemma 4</u> If $F(x)$, $x \in \mathbb{F}_q$ is a one-param. family of complete exponential sums, associate a representation V of $\pi_1(U)$, $U \subseteq \mathbb{A}_{\mathbb{F}_q}$ open

s.t. $F(x) = \mathrm{tr}(\mathrm{Frob}_{q,x} V)$

Theorem of Deligne:

$$\sum_{x\in\mathbb{F}_q} \underbrace{F_1(x)\overline{F_2(x)}}_{\substack{\text{Suitably normalized}\\ F_i \text{ assoc. to } V_i}} = q\,\mathrm{tr}(\mathrm{Frob}_q \ \mathrm{Hom}(V_2,V_1)) + O(\sqrt{q})$$

Rep'ns. $\mathcal{R}_{\lambda,\vec{b}}$ assoc. to R

Lemma 4 follows if they are $\underbrace{\text{irred.}}_{\text{hard part}}$ and distinct (as λ varies)

Remark: ~~Pf of Thm 4 ⇒ Th~~

Lemma 4 ⇒ Thm 1 does not use properties of Kl_k, so possibly room for generalization (though bound in Lemma 4 uses Kl_k).