The Chow Betti numbers of a hypersimplex are ranks of Chow cohomology groups of the torus orbit closure of a generic point in the Grassmannian. They are also dimensions of Minkowski weights on the normal fan of the hypersimplex, which are functions on the set of cones of the fan satisfying a balancing condition. We give explicit formulas for these numbers. We also show that similar formulas hold for the toric h numbers of dual hypersimplices and coordinator numbers of type A^* lattices. This is based on a joint work with Charles Wang.

9/1/17
3:30pm

Chow-Betti Numbers of Hypersimplices
Josephine Yu
(joint work with Charles Wang; preprint on arXiv)

Hypersimplex $\Delta_{k,n}$ $(0<k\leq n)$ is the convex hull of 0/1 vectors in $\mathbb{R}^n$ with exactly k 1's

$$= \{ x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ \sum x_i = k \}$$

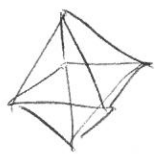
Polytope P

$f_i = \#\, i$-dimensional faces of P

$f(x) = f_{d-1} + f_{d-2}x + \ldots + f_0 x^{d-1} + x^d$

$h(x) = f(x-1) = h_d + h_{d-1}x + \ldots + h_0 x^d$

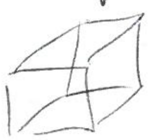
Example: Octahedron



$$f(x) = 8 + 12x + 6x^2 + x^3$$
$$f(x-1) = 1 + 3x + 3x^2 + x^3$$

Theorem: (Dehn-Sommerville) h-vector of a simplicial polytope is palendromic.

Example: Cube (<u>not</u> simplicial)



$$f(x) = 6 + 12x + 8x^2 + x^3$$
$$= 1 - x + 5x^2 + x^3$$

<u>Minkowski weights on a fan</u>

Let $\mathcal{F}$ be a complete (rational) fan. in $\mathbb{R}^n$

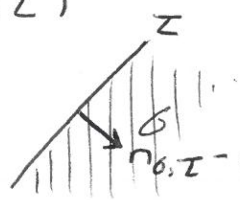
A Minkowski weight for codimension r-cones is a function

$$c : \mathcal{F}^{\text{codim}=r} \longrightarrow \mathbb{R} \quad \text{such that}$$

for every codim $r+1$ cone τ,

$$\sum_{\substack{\sigma \supseteq \tau \\ \sigma \in \text{codim } r}} c(\sigma)\, n_{\sigma,\tau} = 0 \pmod{\operatorname{span}_{\mathbb{Q}} \tau}$$

$n_{\sigma,\tau}$: generator of L_σ / L_τ for lattice version



$n_{\sigma,\tau}$ - unit normal for Euclidean version

Example: $\mathcal{F}$ is the normal fan of a polytope
$c: \sigma \longmapsto \text{vol}(\text{face}_\sigma P)$

Valuation $\Rightarrow$ $c(P) + c(Q) = c(P \cup Q) + c(P \cap Q)$
also

<u>MacMullen 90's, Fulton-Sturmfels</u>

- The Minkowski weights form a graded algebra, graded by codimension of the cone
- This is $\cong$ Chow cohomology ring of toric variety of the fan (if $\mathcal{F}$ is rational)
- If $\mathcal{F}$ is the normal fan of a simple polytope, P, then dim (codim-i Minkowski weights) $= h_i(P^*)$

. This is also a subalgebra of McMullen's polytope algebra

↳ $[P]$ where addition is formal

$$\left.\begin{array}{l} [P] = [P+v] \\ {[P] + [Q] = [P \cup Q] + [P \cap Q]} \text{ is } P \cup Q \text{ is convex} \\ {[P][Q] = [P+Q]} \end{array}\right\} \text{mod out by these relations}$$

The algebra of Minkowski weights $\overset{\text{over all fans}}{\wedge}$ is isomorphic to the polytope algebra (Fulton-Sturmfels '97)

$$\mathrm{vol}^{(r)} : \sigma \mapsto \mathrm{vol}(\mathrm{face}_\sigma P)$$

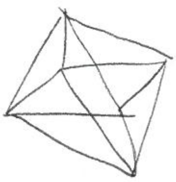
Isomorphism given by $[P] \leftrightarrow \exp(\mathrm{vol}_P^{(\cdot)})$

$$= 1 \oplus \frac{\mathrm{vol}_P^{(1)}}{1} \oplus \frac{\mathrm{vol}_P^{(2)}}{2!} \oplus \dots \oplus \frac{\mathrm{vol}_P^{(n)}}{n!}$$

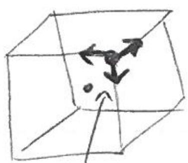
Chow Betti number
β_r = dim of space of Minkowski weights of codim r in the normal fan

Theorem: (Wang - Y. '17) For $1 \le k \le \lfloor \frac{n}{2} \rfloor$

$$\beta_r = \begin{cases} \sum_{i=0}^{r-1} \binom{n}{i} & \text{if } 1 \le r \le k \\ \sum_{i=0}^{k-1} \binom{n}{i} & \text{if } k < r < n-k \\ \sum_{i=0}^{n-r-1} \binom{n}{i} & \text{if } n-k \le r \le n-1 \end{cases}$$



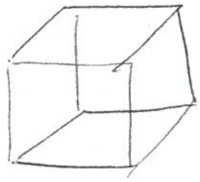
$\Delta_{2,4}$



← Normal fan of $\Delta_{2,4}$ is the face fan of cube

locally the same as

only one way to balance



$\Rightarrow \beta_1 = 1$
$\beta_0 = 1$

$\beta_{r-2} = 2n - (n-1)$
$= n+1$

$$\beta_r = \begin{cases} \sum_{i=0}^{r-1} \binom{n}{i} & \text{if } 1 \le r \le k \\ \sum_{i=0}^{k-1} \binom{n}{i} & \text{if } k < r < n-k \\ \sum_{i=0}^{n-r-1} \binom{n}{i} & \text{if } n-k \le r \le n-1 \end{cases}$$

$\longleftarrow$ difficult linear algebra de Caen 2003

easy

Toric h-numbers of an Eulerian poset, L

- graded with $\hat{0}$ and $\hat{1}$
- $\forall u,v$ with $\text{rank}(v)-\text{rank}(u)=2$, $\mu(u,v)=\pm 1$, $v \neq \hat{1}$

$h(L)$, $g(L)$

$h(\circ) = 1$, $g(\circ) = 1$

$$h(L) = \sum_{\substack{p \in L \\ p \neq \hat{1}}} g([\hat{0}, p])(x-1)^{\text{rk}\hat{1} - \text{rk}p - 1} = \sum_{i \ge 0} h_i x^i$$

$$g(L) = h_0 + (h_1 - h_0)x + \dots + (h_m - h_{m-1})x^m$$

$m = \lfloor \deg h / 2 \rfloor$

(Stanley) Toric h-vectors are palendromic and unimodal

Example: $h(B_{n+1}) = 1 + x + \dots + x^n$
$g(B_{n+1}) = 1$

For face poset of a simplicial polytope,
h = toric-h

(See paper for Chow-Betti numbers and toric h-numbers of many hypersimplices)