Speaker: Dusa McDuff

January 18, 2018
9am, McDuff

Talk Title: An introduction to Symplectic Gromov-Witten Theory.

$M^{2n}, \omega \quad d\omega = 0, \omega^n \neq 0$

J- almost complex structure

$J_x = T_x M \to T_x M, \ J_x^2 = -Id$. Conditions:

- $\omega(v, Jv) > 0$ (if $v \neq 0$) tame
- $\omega(v, w) = \omega(Jv, Jw)$

$\left.\right\}$ compat.

$g_J(v,w) = \omega(v, Jw)$
symm.

ex. Kahler manifold.
complex manifold.
$M, \underset{\substack{\wedge \\ \text{integrable.}}}{J}, g_J \ (d\omega = 0$

- for all $M, \omega$ $\exists$ contractible family of tame (or compact) J.
$(Sp(2n, \mathbb{R}) \simeq U(n))$

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- If $\dim M = 2$, every J is integrable.
not true if $\dim > 2$.
$\nexists$ holom. functions $M^{2n} \to \mathbb{C}^k$ for $n > 1$.
$\exists$ J-holomorphic functions $f: (\underset{\substack{\text{Riemann} \\ \text{surface}}}{\Sigma^2}, j) \longrightarrow (M^{2n}, J)$

quadrics in $\mathbb{C}^2 \subseteq \mathbb{CP}^2$
are solutions $xy = \underset{\substack{\uparrow \\ \text{cmpx}}}{\varepsilon} z^2$

$\mathbb{C}^2 \xrightarrow{f} \mathbb{C}$

$f^{-1}(0)$

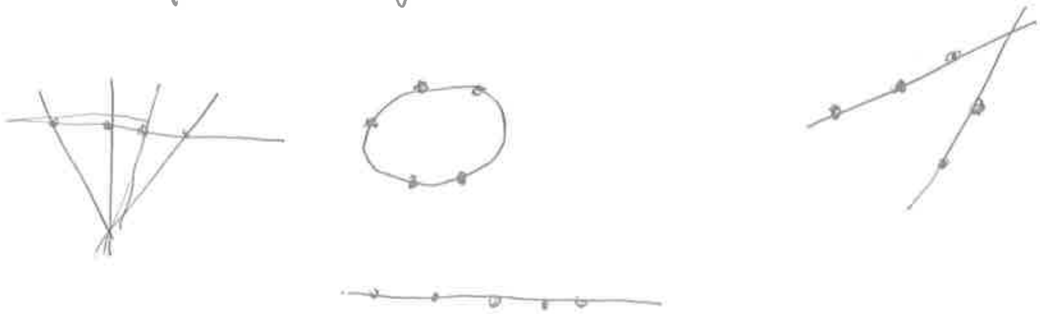
if $C^2 \subset M$
2-dim.
$J: TC \to TC$
$J|_C$ is a complex structure on C
$\therefore \exists f: (\Sigma, j) \xrightarrow{\simeq} (C, J)$

Holom. equation: $\underline{df \circ j = J \circ df}$

<u>EX</u>: $\mathbb{CP}^2 = \mathbb{C}^3\setminus\{0\}/\mathbb{C}^*$, $J_0$ fixed.

• $\exists!\,1$ holo line ($\mathbb{C}$) through any 2 pts.  $\mathbb{CP}^1 = S^2$

• $\exists!$ quadric (deg 2 cone) through 5 generic points



$f:(S^2, j_0) \to (\mathbb{CP}^2, J)$  $\bar\partial_J f = 0$, $\bar\partial_J f = \frac{1}{2}(df + J\circ df\circ j)$

$\quad \| $

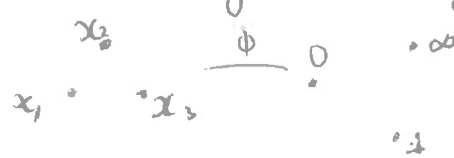
$\quad \mathbb{C}\cup\infty$

count maps $f$ image through the constraints / parametrization

Hol maps $(S^2,j)\circlearrowleft$ = Mobius transf. $\quad PSL(2,\mathbb{C})$

$z\longmapsto \dfrac{az+b}{cz+d}$ $\quad$ 6 real dim, $\begin{pmatrix}a&b\\c&d\end{pmatrix}$

acts triply transitively




$$\left\{ f:(S^2,j)\to(\mathbb{CP}^2,J_0) : \begin{array}{l} f(0)=p_0 \\ f(\infty)=p_1\end{array}\right\}/\mathbb{C}^*$$

symplectic counts need to be <u>independent</u> of choice of $J$.

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$$\left\{ f:(\Sigma, j, \underbrace{z_1,\dots,z_k}_{\text{distinct points}})\to(M,J) \;\middle|\; \begin{array}{l}\bar\partial_J f=0 \\ f(z_i)\in \underset{\text{constraints}}{C_i}\end{array}\right\}/\sim \;\leftarrow \text{parametrizations}$$

$f_*([\Sigma]) = A\in H_2(M)$

call this space $\mathcal{M}^0_{g,k}(M,J,A)$

genus

restrict to genus 0 case.

$(\Sigma,j,z,f)\sim(\Sigma',j',z',f')$ if

$\exists\,\phi:(\Sigma,j,z)\xrightarrow{\phi}(\Sigma',j',z')$

$$\begin{tikzcd} (\Sigma,j,z) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (\Sigma',j',z') \arrow[dl, "f'"] \\ & M & \end{tikzcd}$$

$\mathcal{M}^*(A,J) = \{ f : (S^2, j) \to (M, J) \mid f_*[S_2] = A,\ \bar\partial_J f = 0,\ f \text{ somewhere } ~~i~~\text{njective (injective)} \}$

mapping space

$\exists\, z \in S^2 : f^{-1}(f(z)) = z$

**Theorem 1:**

$\mathcal{M}^*(A, \mathcal{J}^\ell) = \{(f,J) \mid f \in \mathcal{M}^*(A,f)\}$

compact complex structures on M
(large)

is a Banach manifold $\subseteq W^{k,p}(\text{map}) \times \mathcal{J}^\ell$, $k < \ell$.

**Thm 2.** $\mathcal{M}(A, \mathcal{J}^\ell) \xrightarrow{\pi} \mathcal{J}^\ell$ is Fredholm

$(f,J) \longmapsto J$ index $= 2n + 2c_1(A)$

$\mathcal{L}\pi$ : has finite dim. kernel
linear & f. " cokernel

index $= \dim(\ker) - \dim(\text{coker})$

**Thm 3.** Sard-Smale Thm:

$\pi$ is $C^{\ell-k}$-smooth.

$\therefore$ for large $\ell$, $\mathcal{J}^\ell_{reg} = \{ J : \mathcal{L}\pi(f,J) \text{ is onto for all } f \in \pi^{-1}(J) \}$

is residual $= \bigcap_\infty$ open dense sets.

$\Rightarrow \mathcal{M}^*(A,J)$ manifold,

$J \in \mathcal{J}_{reg}$.

In fact, $\mathcal{J}^\infty_{reg}$ ($C^\infty$ reg.) is residual in $\mathcal{J}^\infty$.

**Theorem 4:**

$J_0, J_1 \in \mathcal{J}^\infty_{reg}$. $\mathcal{M}^*(A, \mathcal{J}^\infty)$

$\downarrow$

$J_0, J_1, \mathcal{J}^\infty_{reg} \subseteq \mathcal{J}^\infty$ contr., $J_0$ $J_t$ $J_1$

$\exists\, J_t$ joining them st $\bigcup_t \mathcal{M}^*(A, J_t)$ is a mfld.

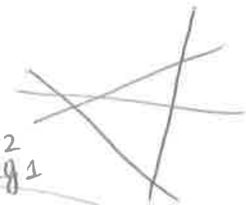
without compactness, cobordism theorem not useful

leads us to the question of compactness.

In good cases $\mathcal{M}^*(A,J)/G = PSL(2,\mathbb{C})$ is <u>compact</u>.

<u>ex</u>: $A = (\text{line})$ in $\mathbb{CP}^2$, $J$

$\mathcal{M}^*(A, J_0)/G$ = space of lines in $\mathbb{CP}^2$



$\boxed{J_0 \text{ regular}}$

$ev : \mathcal{M}^*(A, J_0) \times (S^2)^k / G \longrightarrow \mathbb{CP}^2$ ← $\deg 1$, $k=2$

$(f, z_1, \ldots, z_k) \longmapsto (f(z_1), \ldots, f(z_2))$ well-defined

dimension = index of the corresponding operator.

$4 + 2c_1 A + 2k - 6 = 4k$

$\therefore 4 + 2k = 4k \quad \therefore \boxed{k=2}$

$A$ = line

$c_1(A) = c_1(T\,\text{line}) + c_1(\text{normal } b)$

$= 2 + 1$

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$A = 2(\text{line}) \rightsquigarrow k = 5$

$\mathcal{M}^*(2\,\text{line}, J_0) \times (S^2)^5/G \xrightarrow{ev} (\mathbb{CP}^2)^5$

<u>not</u> compact.

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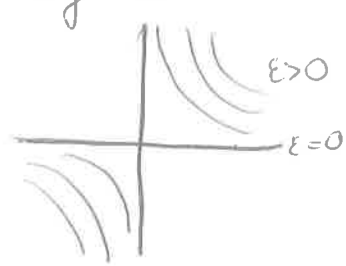
$\mathbb{CP}^1 \qquad\qquad \mathbb{CP}^2$

$[u:v] \xrightarrow{f_\varepsilon} [\varepsilon u^2 : v^2 : uv]$
$\qquad\qquad\quad x \quad\ y \quad\ z$

$xy = \varepsilon z^2$, $\varepsilon \neq 0$

$xy = 0 \qquad \varepsilon = 0$



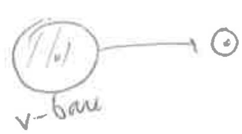

$\varepsilon \to 0$

$[u:v] \longrightarrow [0 : v^2, uv]$ if $v \neq 0$

$= [0 : v : u]$ ~ line $x = 0$

To see what happens near $v = 0$, must reparametrize $f_\varepsilon$. $\phi_\varepsilon : [1:v] \longrightarrow [1 : \varepsilon v]$

Look at




($\varepsilon$ small)

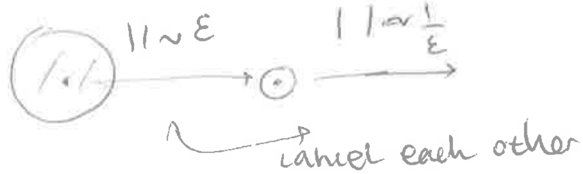
$f \circ \phi_\varepsilon = [1:v] \overset{\phi_\varepsilon}{\longmapsto} [1 : \varepsilon v]$

$\overset{\varepsilon \neq 0}{\longmapsto} [\varepsilon, \varepsilon^2 v^2 : \varepsilon v]$

$= [1 : \varepsilon v : v]$

as $\varepsilon \to 0$ $[1:v] \mapsto [1:0:v]$ - line $y=0$

$|df_\varepsilon(0)| \to \infty$ as $\varepsilon \to 0$.




bubbling phenomenon

Theorem: If $f_i : (\Sigma, j) \to (M, J)$ with $\|df_j\|_{L^2} < \infty$ energy

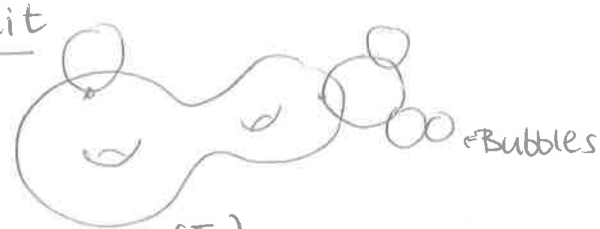
then $\exists$ subsequence (called $f_i$) and finite set $Z \subset \Sigma$

s.t $f_i$ converges uniformly with all derivatives on compact subsets of $\Sigma \setminus Z$.

limit $f_\infty$ is J-hol. $(\Sigma - Z) \to (M, J)$

$\rightsquigarrow f_\infty : (\Sigma, j) \to (M, J)$

image of the limit

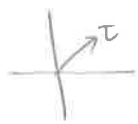
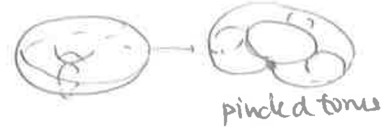



$f_\infty(\Sigma)$

Bubbling - only source of non-compactness for spheres.

- not in higher genus case, since $(\Sigma, j_i) \rightsquigarrow$ can degenerate

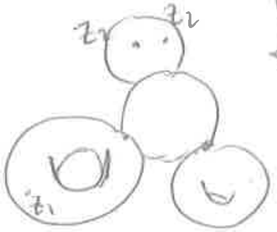
Cubic curves in $(\mathbb{CP}^2, J_0)$ have genus 1.

$(T^2, j) = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$  $\tau \in \mathbb{H}/PSL(2,\mathbb{Z})$ 


Two issues:

- Regularity: ① How do we deal w/ multiply-covered curves?
- Compactness ✓ (stable maps ...)

$\Sigma, j$ nodal Riemann surface + $\underline{z} = z_1, \dots, z_k$ marked points




$$\left\{ (\Sigma, j, \underline{z}, f) : f : (\Sigma, j) \to (M, J) \right\} \Big/ \sim$$

$f$ J-holom. repres. $A$

$(\Sigma, j, \underline{z}, f)$
$\downarrow$
$(\Sigma', j', z', f')$

self-aut. of $(\Sigma, j, \underline{z}, f)$ <u>finite</u>

$\downarrow$

$g(\text{comp.}) = 1$ need $\geq 1$ special pt.

$g(\text{comp.}) = 0$ need $\geq 3$ sp. pts.

$\| \quad \overline{\mathcal{M}}_{g,k}(M, A, J)$

(ghost) ~~ghost~~ comp where $f$ is constant.

<u>Theorem</u>: $\overline{\mathcal{M}}_{g,k}(M, A, J)$ compact.

$\downarrow$ stratified space. codim. of strata $= 2(\#\text{nodes})$

$$X = \overline{\mathcal{M}}_{g,k}(M, A, J) \xrightarrow{\text{eval}} M^k$$

$$f \longmapsto (f(z_1), \dots, f(z_k))$$

$[\Sigma, j, z, f]$

$\downarrow$ stab.

$[\Sigma, j, z]$

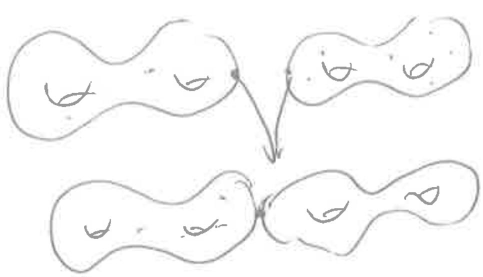
$\overline{\mathcal{M}}_{g,k}$

Deligne-Mumford space

G-W invariants $\quad \langle a_1, \dots, a_k ; \beta \rangle_{g,A} = \int_{[X]^{vir}} ev^*(a_1 \cup \dots \cup a_k) \cup st^*(\beta)$

$a_i \in H^*(M) \quad \beta \in H^*(\overline{\mathcal{M}}_{g,k})$

$$\overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2, k_1+k_2}$$



"concatination"

$\rightsquigarrow$ can define "quantum cup product".