Classical McKay Correspondence

$G \subset SU(2)$ finite

$\pi : Y_G \longrightarrow \mathbb{C}^2/G$ minimal resolution (Y_G is a CY surface)

$\pi^{-1}(0)$ is a configuration of curves $C_1, \cdots, C_n \cong \mathbb{P}^1$ meeting in nodes.

Dual Graph of Exception Divisor $\overset{\cong}{\longleftrightarrow}$ McKay Graph of G

vertices:	$C_1, \cdots, C_n$ exceptional curves	$\rho_1, \cdots, \rho_n$ non-triv. irreducible reprs.
edges	edge connecting C_i & C_j $\iff$ $C_i \cap C_j \neq \phi$	edge connecting ρ_i & ρ_j $\iff$ $\rho_i \leq \rho_j \otimes \mathbb{C}^2$

$\mathbb{C}^2$: rep induced by $G < SU(2)$

example: $G = \{\pm 1, \pm i, \pm j, \pm k\}$ Quaternion 8-gp embed in $SU(2)$ via pauli matrices.

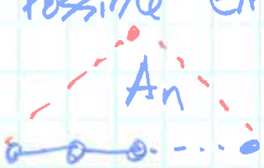
$\pi^{-1}(0) =$ 

$\mathbb{C}^2$ rep

1-dim'l reps pulled back from ~~$SU(2)/\pm$~~ $G/\pm 1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

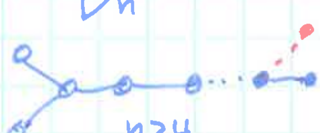
Possible Graphs with n nodes — ADE dynkin diagrams

A_n



$\mathbb{Z}_{n+1}$ $n \geq 1$

D_n



$n \geq 4$

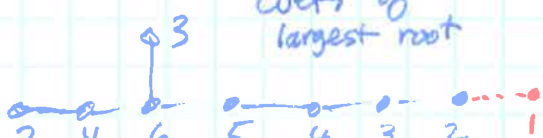
Binary Dihedral

E_6 E_7 E_8

Binary $\begin{cases} \text{tetrahedral} \\ \text{octahedral} \\ \text{icosahedral} \end{cases}$

multiplicities of $\pi^{-1}(0)$ as a scheme

coefs of largest root



dimensions of the representations

To view the representation theory side geometrically use the <u>orbifold quotient</u> $\left[\mathbb{C}^2/G\right]$ smooth DM stack

<u>not a space</u> But we can do geometry on it as if it were a space. ("I can't believe it's not a space!")

Slogan: "Geometry on $\left[Z/G\right]$ is G-equivariant geometry on Z"

$\Rightarrow$ A sheaf on $\left[Z/G\right]$ is a G-equivariant sheaf $\mathcal{F}$ on Z

i.e. $\Phi_g : \mathcal{F} \xrightarrow{\cong} g^*\mathcal{F}$ compatible with group multiplication.

<u>ex</u> $p_0 \in \mathbb{C}^2$ origin $\mathcal{O}_{p_0} \otimes \mathbb{C}^d$ is G invariant, choice of Φ_g defines G action on $\mathbb{C}^d$, i.e. $\mathcal{O}_{p_0} \otimes \underset{G\text{ rep}}{\rho}$ is a sheaf on $\left[\mathbb{C}^2/G\right]$

$\operatorname{Rep} G \cong$ ring generated by $\langle \mathcal{O}_{p_0}, \mathcal{O}_{p_0}\otimes\rho_1, \cdots \mathcal{O}_{p_0}\otimes\rho_n \rangle$ $= K_0^{cpt}\left(\left[\mathbb{C}^2/G\right]\right)$ $\longleftrightarrow$ ring generated by $\langle \mathcal{O}_{C_1}(-1), \cdots, \mathcal{O}_{C_n}(-1), \mathcal{O}_{pt} \rangle$ $K_0^{cpt}(Y_G)$

<u>Launching Point</u> The above raises several natural questions which lead us to the subject of these talks.

Questions

- Geometric Meaning of isomorphism? $\Rightarrow$ Fourier-Mukai
- Cohomology instead of K-theory? $\Rightarrow$ Need QH^*_{orb} $\Rightarrow$ GW theory
- Categorify? Sets $\longrightarrow$ Vector Spaces $\longrightarrow$ Categories

  (Intersection Graph $\rightarrow$ K-theory $\rightarrow$ Derived categories
  (Donaldson-Thomas theory))
- Generalize to other orbifolds, other resolutions? $\Rightarrow$ Crepant Resolution Conjectures (starts with Ruan)

$[\mathbb{C}^2/G] \rightsquigarrow \mathcal{X}$ CY orbifold

$\mathbb{C}^2/G \rightsquigarrow X$ associated singular space

$Y_G \rightsquigarrow Y$ CY resolution of X

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "p"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

more generally include NC resolutions and different resolutions

Seek equivalences:

- Derived Categories of $\mathcal{X}$ & Y (BKR derived generalize McKay corr.)
- Gromov-Witten theory of $\mathcal{X}$ & Y (GW CRC) Physics Ruan
- Donaldson-Thomas theory of $\mathcal{X}$ & Y (DT CRC)
- Elliptic Genera of $\mathcal{X}$ & Y (CRC for Ell Gen Libgober-Borisov)

  ⋮

$$\begin{tikzcd} DT(Y) \arrow[r, dash, "MNOP"] \arrow[d, dash, "DT\ CRC"'] & GW(Y) \arrow[d, dash, "GW\ CRC"] \\ DT(\mathcal{X}) \arrow[r, "\text{orbi-MNOP}"'] & GW(\mathcal{X}) \end{tikzcd}$$

CY3

GW theory of $\mathcal{X} = [Z/G]$ a map $f: C \to \mathcal{X}$ should correspond to a G-equivariant map $\tilde{f}: \tilde{C} \to Z$

The correspondence is via a fiber square:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ C \arrow[r, "f"] & {[Z/G]} \end{tikzcd}$$

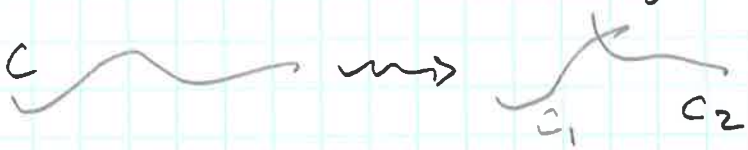
principal G-bundle

principal G-bundle by def'n

categorically $[Z/G]$ behaves like a free quotient.

<u>example</u> $BG = [pt/G]$ $\quad Map(C, BG) = \{ \tilde{C} \to C \text{ principal G-bundle} \}$

In GW theory we need to be able to degenerate C to a nodal curve C



In such a limit, the cover $\tilde{C} \rightsquigarrow \tilde{C}_1 \cup \tilde{C}_2$ no longer free action

We allow C to have orbifold pts.

Any G equivariant map $\tilde{f}: \tilde{C} \to Z$ gives

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d] & Z \arrow[d] \\ \mathcal{C} = {[\tilde{C}/G]} \arrow[r, "f"] & {[Z/G]} \end{tikzcd}$$

$\mathcal{C}$ is an orbi-curve twisted stable map $f: \mathcal{C} \to [Z/G]$ $\quad \mathcal{C}$ can have orbifold points at marked pts and/or nodes.

Example $\mathcal{X} = [\mathbb{C}^2/\pm 1]$ $\quad Y = tot(T^*\mathbb{P}^1)$

genus 0 GW theory

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{C}^2 \arrow[d] \\ \mathbb{P}^1 \arrow[r, "f"] & {[\mathbb{C}^2/\pm 1]} \end{tikzcd}$$

$\tilde{f}$ must map to $(0,0)$

$\tilde{C} \to \mathbb{P}^1$ hyperelliptic curve ramified over $2g+2$ pts no non-zero degree (maps are constant), instead we keep track of # of orbifold pts.

Hodge Classes

$$GW_{0,n}(\mathcal{X}) = \int_{[\overline{M}_{0,n}(\mathbb{C}^2/\pm 1)]^{\text{red vir.}}} 1 = \int_{\overline{H}_g} \lambda_g \lambda_{g-1}$$

n orbifold pts ($B\mathbb{Z}_2$)

$2g+2=n$

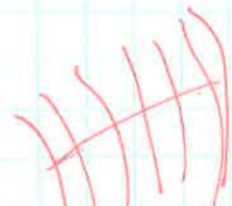
compactification of $H_g \subset M_g$ hyperelliptic locus

Genus 0 GW potential of $\mathcal{X} = [\mathbb{C}^2/\pm 1]$:

$$F^0(\mathcal{X}) = {\sum_n}' GW_{0,n}(\mathcal{X})\, t^n$$

Faber-Pand.

$$\left(\frac{d}{dt}\right)^3 F^0(\mathcal{X}) = +\frac{1}{2}\tan\left(\frac{t}{2}\right)$$



$\mathbb{P}^1$

Genus 0 GW potential of $Y = \mathrm{Tot}(T^*\mathbb{P}^1)$:

maps have degree (class $d[\mathbb{P}^1]$), no interesting insertions

$$GW_{0,d}(Y) = \int_{[\overline{M}_{0,0}(Y,\, d[\mathbb{P}^1])]^{\text{red vir}}} 1 = +\frac{1}{d^3}$$

$$F^0(Y) = \left[\begin{matrix}\text{low}\\ \text{degree} \cdots\\ \text{other variables}\end{matrix}\right] + \sum_{d=1}^{\infty} +\frac{1}{d^3} q^d$$

involving const maps

$$\left(q\frac{d}{dq}\right)^3 F^0(Y) = +\frac{1}{2} + \sum_{d=1}^{\infty} q^d = +\frac{1}{2} + \frac{q}{1-q} = +\frac{1}{2}\,\frac{1+q}{1-q}$$

Let $q = -e^{it}$ $\qquad \left(q\frac{d}{dq}\right)^3 F^0(Y) = +\frac{1}{2}\,\frac{1-e^{it}}{1+e^{it}} = \frac{-1}{2i}\tan\left(\frac{t}{2}\right)$

$$q\frac{d}{dq} = -i\frac{d}{dt} \quad \Rightarrow \quad \boxed{F^0(Y) = F^0(\mathcal{X})}$$

after $q = -e^{it}$ and analytic continuation.

Gromov-Witten Crepant Resolution Conj. (Hard Lefschetz case)

If $\mathcal{X}$ is an orbifold satisfying HL, $Y \to X$ crepant resolution of singular space. Then there exists a change of variables such that the genus g GW-potentials are equal after analytic continuation

$$F^g(\mathcal{X}) = F^g(Y)$$

- General non-HL case must be phrased in terms of Givental's Lagrangian cone formalism. (Coates Corti Iritani Tseng, Ruan).