INFINITESIMAL DEFORMATIONS OF VARIETIES WITH TRANSVERSAL RPDS

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(*) Notes taken by Dhyan Aranha, all errors should be attributed to me and my ignorance about the subject. Corrections and suggestions are welcome, and should be sent to: dhyan.aranha@gmail.com.

The talk is about joint work in progress with Alex Massarenti. We will work over $\mathbb{C}$. Also in case you were wondering: RDP - Rational Double Point.

Definition 0.0.1. *A surface S has $p \in S$ as RDP if étale locally it is isomorphic to $\mathbb{C}^2/G$, for $G \subseteq SL(2, \mathbb{C})$ finite subgroup.*

RDP's are classified and they are of the following types:

$$(A_n)_{n\geq 1}, \qquad (D_n)_{n\geq 4}, \qquad E_6, \qquad E_7, \qquad E_8.$$

If our surface S has a RDP then it also has a minimal resolution

$$\epsilon_S : \tilde{S} \longrightarrow S$$

with exceptional divisor with simple normal crossings. Moreover S is Gorenstein and ϵ_S is a crepant resolution so the canonical line bundle on $\tilde{S}$ is the pullback of the dualizing sheaf on S. The problem of relating the infinitesimal deformations of S with those of the resolution has been studied extensively in the 70's by a number of people to name a few: Artin, Brieskorn, Burns, Wall, etc...

Let X be a variety such that $Sing(X)$ is non-singular closed subvariety, and such that (X, Z) is étale locally isomorphic to $(S \times Z, Z)$ with S a fixed $\mathbb{C}^2/G$.

Remark 0.0.2. *One could think of a variety having multiple transversal loci, but for today we will focus on the case when there is only one.*

To X we can associate two non-singular objects

$$\begin{array}{ccc} \mathcal{X} & & Y \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

Where $f : Y \longrightarrow X$ is the minimal resolution of the singularity. The other one $\epsilon : \mathcal{X} \longrightarrow X$ is the canonical smooth Deligne-Mumford stack with X as its coarse moduli space. Both maps are crepant and

$$f^*\omega_X = K_Y, \quad \epsilon_X^* = K_{\mathcal{X}}.$$

Let's recall what deformation functors are: Consider the category

$$(Art) = \{\text{local Artinian finitely generated } \mathbb{C}\text{-algebras with residue field } \mathbb{C}\}.$$

Lemma 0.0.3. *$A \in (Art) \iff \mathrm{Spec}(A)$ is a scheme of finite type over $\mathbb{C}$ such that $\mathrm{Spec}(A)_{red} = \mathrm{Spec}(\mathbb{C})$.*

Definition 0.0.4. *Let $Def_X : (Art) \longrightarrow (Set)$ be the functor which sends a $A \in (Art)$ to the collection of diagrams of the form*

$$\begin{array}{ccc} X & \longrightarrow & X_A \\ \downarrow & \square & \downarrow \scriptstyle{flat} \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

modulo isomorphism.

Notice: If $U \subseteq X$ open, then we get a functor $Def_X \longrightarrow Def_U$. This is essentially because the map $X \longrightarrow X_A$ is a homeomorphism.

Definition 0.0.5. *We say a deformation is trivial if it is isomorphic to $X \times \mathrm{Spec}(A)$.*

Remark 0.0.6. *(Dhyan) I think when people refer to a specific deformation as we did in the above definition it simply means a flat morphism $Y \longrightarrow \mathrm{Spec}(A)$ such that there exists a cartesian diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \square & \downarrow \scriptstyle{flat} \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A). \end{array}$$

Now we introduce a subfunctor of $LTDef_X \subseteq Def_X$ the so called, locally trivial deformations functor. It means that: there is an affine Zariski open cover of X on which the induced deformation becomes trivial.

For smooth var/ orbifolds/ DM-stacks - all deformations are locally trivial but this is **not** the case for singular varieties.

We have the following diagram

$$\begin{array}{ccccc} & & LTDef_X & \hookrightarrow & Def_X \\ & \swarrow \simeq & \downarrow \simeq & & \uparrow \\ Def_{\mathcal{X}} & & Def_{(Y,E)} & \hookrightarrow & Def_Y \end{array}$$

where $E \subseteq Y$ is the exceptional divisor i.e. $E = f^{-1}(Z)_{red}$, it is in general a normal crossing divisor.

Let's say a few words about how the map $Def_Y \longrightarrow Def_X$ is defined: If you have a deformation Y_A this is the same thing as giving a sheaf of flat A-algebras, $\mathcal{O}_{Y_A}$, on the topological space Y plus a surjection $r : \mathcal{O}_{Y_A} \twoheadrightarrow \mathcal{O}_Y$ with a few good local properties. Thus if you have a deformation of Y in this way, you just push-forward the map r via f to get a map $f_*\mathcal{O}_{Y_A} \longrightarrow \mathcal{O}_X$.

<u>Question:</u> When is $LTDef_X \simeq Def_X$? Similarly when is $LTDef_X \simeq Def_Y$?

Definition 0.0.7. *$F : (Art) \longrightarrow (Set)$ is a deformation functor with tangent-obstructions T^1F, and T^2F. If T^1F and T^2F are vector spaces over $\mathbb{C}$, and*

for every surjection in (Art), $A \twoheadrightarrow B$ *with kernel* I *such that* $\mathfrak{m}_A I = 0$ *we have an exact sequence*

$$T^1 F \otimes_C I \longrightarrow F(A) \longrightarrow F(B) \longrightarrow T^2 F \otimes_C I$$

which is functorial and exact on the left if $\mathfrak{m}_A^2 = 0$.

Our functors Def and $LTDef$ have tangent spaces and they are:

$$T^i Def_X = Ext^i(\Omega_X, \mathcal{O}_X)$$
$$T^i LTDef_X = H^i(X, T_X).$$

Criterion: Given $\alpha : F \longrightarrow G$ of deformation functors. If α induces an isomorphism on T^1 and is injective on T^2 then it is an equivalence. (proved via induction on the vanishing of the power of the maximal ideal)

We now recall 2 useful spectral sequences:

i) (Local to global for Ext)

$$H^q(\mathscr{E}xt^p) \Rightarrow Ext^{p+q}.$$

ii) (Leray)

$$H^q(R^p f_*) \Rightarrow H^{p+q}.$$

These give the exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow H^1(T_X) & \longrightarrow & Ext^1(\Omega_X, \mathcal{O}_X) & \longrightarrow & H^0(\mathscr{E}xt^1) & \longrightarrow & H^2(T_X) & \longrightarrow & Ext^2(\Omega_X, \mathcal{O}_X) \\ \uparrow \scriptstyle{\text{id}} & & \uparrow & & \uparrow \scriptstyle{0} & & \uparrow \scriptstyle{\text{id}} & & \uparrow \\ 0 \longrightarrow H^1(T_X) & \longrightarrow & H^1(T_Y) & \longrightarrow & H^0(R^1 f_* T_Y) & \longrightarrow & H^2(T_X) & \longrightarrow & H^2(T_Y) \end{array}$$

It's an easy fact to check that $T_X = f_* T_Y$.

Concrete Aim: Compute $\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)$ and $R^1 f_* T_Y$ as coherent sheaves set theoretically supported on Z.

If $Z = pt$, $R^1 f_* T_Y \cong \mathcal{O}_Z^{\oplus n}$, $\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)$ is an invertible sheaf on $Z_n \subseteq X$ closed sub-scheme such that $(Z_n)_{red} = Z$ and $length Z_n = n$. This implies that $h^0(R^1 f_* T_Y) = h^0(\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)) = n$.

In the general case $R^1 f_* T_Y$ is a rank n loc. free sheaf on Z and $\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)$ is a line bundle on a closed sub-scheme $Z_n \subseteq X$ such that $(Z_n)_{red} = Z$ and $[Z_n] = n[Z]$.

In order to do the computation there is a useful trick: Degenerate $\mathcal{Z} \hookrightarrow \mathcal{X}$, $f = \epsilon(Z)_{red}$ into $\mathcal{Z} \hookrightarrow C_{\mathcal{Z}/\mathcal{X}} = \mathcal{N} := \mathcal{N}_{\mathcal{Z}/\mathcal{X}}$. This gives us induced degenerations:

$$Z \hookrightarrow X \quad to \quad Z \hookrightarrow X_0 = \text{Coarse space of } \mathcal{N}$$

once of have this you can take the minimal resolution:

$$\begin{array}{ccc} & & Y_0 \\ & & \downarrow{\scriptstyle min\ res} \\ Z & \longrightarrow & X_0. \end{array}$$

<u>A_1:</u> Because line bundles don't degenerate (since the picard scheme is separated) you can compute $\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)$ and $R^1 f_* T_Y$ as $\mathscr{E}xt^1(\Omega_{X_0}, \mathcal{O}_{X_0})$ and $R^1 f_{0*} T_{Y_0}$.

What is the advantage of this? Well you have much simpler situation because $\mathcal{N}$ is a rank two bundle on a gerbe $\mathcal{Z}$, On this gerbe every point has an automorphism group, G, and so the group acts on the fibers on the bundle. In particular we have a map

$$\epsilon_Z : \mathcal{Z} \longrightarrow Z$$

so we can pull back sheaves on Z to the gerbe, and if we have a sheaf on $\mathcal{Z}$ how do we recognize if it's a pullback? The answer is: if and only if the action of the group G is trivial.

Now if you look at $\mathcal{N}$, remember that our group acted via SL_2. So there is certainly one line bundle which comes from Z and that is the determinant of $\mathcal{N}$. i.e. There exists a unique $L \in Pic(Z)$ such that $\epsilon_Z^* L = det\mathcal{N}$.

Lemma 0.0.8. *(Fantechi-Massarenti) In the A_1 case*

$$\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X) \simeq L^{\otimes 2}$$

and

$$R^1 f_* T_Y = L.$$

What can we say in the general case? Well let's say something first about the exceptional divisor E.Consider the map $E \longrightarrow Z$, étale locally in Z this is a product of a configuration of curves indexed by the Dynkin diagram.

We will say that we are in an <u>Easy</u> situation if the action of the fundamental group of Z on the Dynkin diagram is trivial. (This is true in particular if Z is simply connected)

Theorem 0.0.9. *(Fantechi-Massarenti) Let X have <u>easy</u> transversal RPD along Z (again: what we mean by easy is that fundamental group of Z acts trivially on the Dynkin diagram). Then*

$$R^1 f_* T_Y \cong \bigoplus_{i=1}^{n} L.$$

Remark 0.0.10. *In the non-easy case, you get induced $Z' \longrightarrow Z$ un-ramified cover. For example if the cover is (2:1) then $\pi_* \mathcal{O}_{Z'} = \mathcal{O}_Z \oplus M$, and $M^{\otimes 2} \cong \mathcal{O}_Z$ (i.e. it is 2-torsion). Then what you get is that $R^1 f_* T_Y$ has summands $L, L \otimes M$.*

Theorem 0.0.11. *(Fantechi-Massarenti) In the A_n easy case $\mathscr{E}xt^1(\Omega_X, \mathcal{O}_X)$ has a natural filtration with associated graded $\bigoplus_{i=2}^{n+1} L^{\otimes i}$*

Expectation: In easy D_n case there exists a filtration and all summands are $L^{\otimes i}$ for $2 \leq i \leq n$.

Sketch of proof: First you prove there exists a natural filtration with line bundle quotients.(Aside: in the A_n case this is really easy because locally your equation is $xy = z^{n+1}$ so you have $Z_n = (x, y, z^n)$ so the filtration is given by just taking lower and lower powers of z). Once you have them then you can go to do the degeneration.

For the other case you take $\nu : \tilde{E} \longrightarrow E$ be the normalization of exceptional divisor.

$$0 \longrightarrow T_{\tilde{E}} \longrightarrow \nu^* T_Y \longrightarrow N_\nu \longrightarrow 0$$

On $\tilde{E}$, which gives

$$T_Y \longrightarrow \nu_* N_\nu$$

which gives a map

$$R^1 f_* T_Y \longrightarrow R^1 f_* \nu_* N_\nu$$

Burns-Wall: The map $R^1 f_* T_Y \longrightarrow R^1 f_* \nu_* N_\nu$ is an isomorphism in the surface case. But once is true in the surface case its true in the product case.

If you are in the easy case $\implies$ $E_1, \ldots, E_n$ irreducible components of E are non singular and $\nu_* N_\nu = \bigoplus_{i=1}^n N_{E_i/Y}$. This implies that $R^1 f_* T_Y \cong \bigoplus_{i=1}^n R^1 f_* N_{E_i/Y}$ where $R^1 f_* N_{E_i/Y}$ are line bundles. Now you can use that f is crepent. (i.e. you play around with exact sequences and you get something like $R^1 f_* \Omega_{E_i/Z}$ of course this is trivial, with this you do the computation)

Recall: L was defined as the only line bundle on Z such that $\epsilon_Z^* L \cong \det \mathcal{N}$. Remember though that $\mathcal{N} = \mathcal{N}_{\mathcal{Z}/\mathcal{X}}$, now you have that $\det \mathcal{N} = K_{\mathcal{Z}} \otimes K_{\mathcal{X}}^\vee|_{\mathcal{Z}} = \epsilon_Z^* K_Z \otimes \epsilon_Z^* (\omega_X^\vee|_Z)$ it follows that $L \cong K_Z \otimes \omega_X^\vee|Z$.

Now consider $\overline{\mathcal{M}}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$ for $2g-2+n > 0$. Then yo know that $Def_{\overline{\mathcal{M}}_{g,n}}$ is trivial (Hacking) and $LTDef_{\overline{M}_{g,n}}$ trivial (Hacking). How about without ”LT”?

Theorem 0.0.12. *(Fantechi-Massarenti)* $LTDef_{\overline{M}_{g,n}} = Def_{\overline{M}_{g,n}}$ *unless* $(g, n) = (1, 2)$.