Character ratios for finite groups of Lie type

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A **character ratio** for \( G \) a finite group is \( \frac{\chi(g)}{\chi(1)} \) for \( \chi \in \text{Irr}(G) \) or \( \text{IBr}(G) \).

Applications of character ratios come via: If \( C_1, \ldots, C_d \) are conjugacy classes in \( G \), the number of solutions \((x_1, \ldots, x_d)\) to \( x_1 \cdots x_d = z \) for \( x_i \in C_i \) is

\[
\prod \frac{|C_i|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(c_1) \cdots \chi(c_d) \overline{\chi(x)}}{\chi(1)^{d-1}},
\]

where \( c_i \in C_i \), a classical result going as far back as Frobenius.

1 **Applications**

1) Counting points in representation varieties

\[ \text{Hom}(\Gamma, G), \]

for \( \Gamma \) finitely presented.

**Example**

\[ \Gamma = T_{abc} = \langle x, y, z \mid x^a = y^b = z^c = xyz = 1 \rangle. \]

Count solutions to equation (1) with \( z = 1 \) over classes of order \( a, b, \) and \( c \).

2) Random walks:

\[ G = (C), \quad C = x^G. \]

We look at a random walk

\[ 1 \to c_1 \to c_1 c_2 \to \cdots \]

This is a Markov chain with eigenvalues given by character ratios \( \frac{\chi(c)}{\chi(1)} \) for \( \chi \in \text{Irr}(G) \).

\[ P_k(g) = \text{probability at } g \text{ after } k \text{ steps.} \]

Usually \( P_k \to U \). How fast?

**Diaconis-Shahshahani**:

\[
||P_k - U||^2 = \left( \sum_{g \in G} |P_k(g) - U(g)| \right)^2 \leq \sum_{\chi \neq 1} \left| \frac{\chi(x)}{\chi(1)} \right|^{2k} \chi(1)^2.
\]
3) McKay graphs:

For $G$ a finite group, $\alpha$ a character, we define a graph

$$\Gamma(G, \alpha)$$

with vertices given by $\text{Irr}(G)$, and directed edges $\chi \to$ constituents of $\chi \otimes \alpha$.

**Example** 1) $G = C_n$, $\alpha$ linear character generator:

2) $G = SL_2(5)$, $\alpha$ having degree 2:

3) $G = SL_2(p)$, $\alpha$ = 2-dimensional $\mathbb{F}_p^2$ natural module:

These are called McKay graphs due to the **McKay correspondence**: For $G$ a finite subgroup of $SU_2(\mathbb{C})$, and $\alpha$ a 2-dimensional representation, we have

$$\Gamma(G, \alpha) = \tilde{A}, \tilde{D}, \tilde{E}.$$ 

**Theorem 1.1** (Burnside-Brauer) If $\alpha$ is faithful, then every $\chi \in \text{Irr}(G)$ appears in $\alpha \otimes n$ for some $n \leq \#\{\alpha(g) : g \in G\} / N$.

Define $\text{diam}(G, \alpha) = \text{diam}(\Gamma(G, \alpha)) \leq 2N$. Clearly

$$\text{diam}(G, \alpha) \geq \frac{\log(\text{maximal degree})}{\log \alpha(1)}.$$ 

**Example** For $G = S_n$, $\alpha = \chi^{(n-1,1)}$: we have $n \geq \text{diam} \geq \frac{n}{4}$.

For $G = G(q)$, $\alpha = St$ Steinberg character: $\text{diam}(G, St) = 2$ with one exception when $G = U_n(q)$ (Heide-Saxl-Tiep-Zalesski).

## 2 Results

**Theorem 2.1** (Gluck) For $G = G(q)$, $\chi \in \text{Irr}(G)$,

$$\frac{|\chi(g)|}{\chi(1)} < \frac{3}{\sqrt{q}}.$$
The setting for the next result by Bezrukavnikov-Liebeck-Shalev-Tiep (2016) is: If \( G = G(q) = G^F \) for \( G \) a simple algebraic group, and a Levi \( L \) of \( G \), define

\[
\alpha(L) = \max \left( \frac{\dim u^L}{\dim u^G} : u \neq 1 \text{ unipotent} \right)
\]

where \( u^L \) denotes the conjugacy class of \( L \), etc.

**Example** If \( G = SL_3 \) and

\[
L = \begin{pmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & *
\end{pmatrix}
= GL_2,
\]

then \( \alpha(L) = \frac{2}{4} = \frac{1}{2} \).

We have \( \alpha(T) = 0 \) for \( T \) a torus.

Say \( L \) is split Levi if \( L^F \leq P^F \), with \( P \) parabolic.

**Theorem 2.2** (Bezrukavnikov-Liebeck-Shalev-Tiep 2016)

Suppose \( G = G(q) \) \( (p \text{ a good prime}) \) is simply connected. Let \( x \in G \) and suppose \( C_G(x) \leq L^F \), split Levi. Then for all \( \chi \in \text{Irr}(G) \)

\[
\chi(x) < \chi(1)^{\alpha(L)} \cdot f(r)
\]

where \( r = \text{rk}(G) \).

For \( G = SL_n, f \sim n! \).

**Example** 1) \( G = SL_3(q) \), the theorem applies to all \( x \) except unipotent elements and regular semisimple elements with centralizer order \( q^2 + q + 1 \).

For the remaining elements, we have

\[
\left| \frac{\chi(x)}{\chi(1)} \right| < \chi(1)^{-\frac{1}{2}} \cdot c.
\]

2) For \( G = GL_n(q) \):

\[
L = \prod_{i=1}^{t} GL_{n_i}(q) \quad n_1 \geq n_2 \geq \cdots
\]

we have

\[
\frac{n_1 - 1}{n - 1} \leq \alpha(L) \leq \frac{n_1}{n}
\]

3) \( G = E_8(q) \)

\[
\begin{array}{c|cccc}
\alpha(L) & E_7 & D_7 & \cdots & \text{most} \\
\hline
L & \frac{14}{29} & \frac{14}{23} & \leq \frac{14}{23}
\end{array}
\]

### 3 Random Walk on E8(q)

For \( G = E_8(q) \), for \( x \in G, C_G(x) \) contained in a split Levi

\[
\|P_k - U\|^2 \leq \sum_{x \neq 1} \left| \frac{\chi(x)}{\chi(1)} \right|^{2k} \chi(1)^2 \leq \sum \chi(1)^{2k(-1+\alpha)+2}
\]
For $\alpha = \frac{17}{29}$, $k = 3$, this equals
$$\sum_{\chi \neq 1} \chi(1)^{-2/29} \to 0.$$  

Liebeck-Shalev:
$$\sum_{\chi \in G(q)} \chi(1)^{-s} \to 1, \quad s > \frac{2}{h}.$$  

For $E_{8}$, $h$ is equal to 30, hence
$$\text{Mix}(E_{8}(q), x^{G}) \leq 3.$$  

4 Remaining results

Liebeck-Shalev-Tiep: $G = SL_{n}(q), x \in G$.
Define $s =$ codimension of largest eigenspace of $x$ over $\mathbb{F}_{p}$.

Example Say $x$ is unipotent, a sum of $t$ Jordan blocks,
$$x = \sum_{i=1}^{t} J_{n_{i}}, \quad s = n - t.$$  

Theorem 4.1 For all $\chi \in \text{Irr}(G)$,
$$\frac{|\chi(x)|}{\chi(1)} \leq \frac{1}{q^{\gamma s}} f(n),$$
with $\gamma \approx \frac{1}{9}$.

Recall $G$ simple, $\alpha \in \text{Irr}(G)$,
$$\text{diam}(G, \alpha) = \min(k : \text{Irr}(G) \subset \alpha \cup \cdots \cup \alpha^{k}).$$

We define
$$D(G) = \max_{\alpha} \text{diam}(G, \alpha).$$

Theorem 4.2 (Liebeck-Shalev)
For $C$ a conjugacy class of $G$, $\text{diam}(G, C) \leq \beta \frac{\log |G|}{\log |C|}.$

Conjecture 4.3
$$\text{diam}(G, \alpha) \leq \delta \frac{\log |G|}{\log \alpha(1)}.$$  

Theorem 4.4 For $G = SL_{n}(q)$, $D(G) \leq cn$ provided $q > f(n)$ (here $c \sim 50$).

Proof Know $\text{Irr}(G) \subset S_{t}^{2}$.
So we aim to show $S_{t} \subseteq \chi^{cn}$ for all $\chi \in \text{Irr}(G)$.
$$\langle \chi^{\ell}, S_{t} \rangle = \frac{1}{|G|} \sum_{g \in G, s} \pm \chi^{\ell}(g) C_{G}(g)|_{p} = \frac{\chi^{\ell}(1)}{|G|} \sum_{g \neq 1} \left( |G|_{p} + \sum_{g \neq 1} \frac{\chi^{\ell}(g)}{\chi^{\ell}(1)} |C_{G}(g)|_{p} \right).$$

Now use the bound for the character ratios $\frac{\chi(g)}{\chi(1)}$. 

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