On equivariant cohomology of Calogero-Moser spaces

Lecture by Peng Shan
Notes by Dustan Levenstein

Joint with Cédric Bonnafé.

1 Calegero Moser Spaces

Let W be a finite complex reflection group, with reflection representation V. Let S be the generating set of pseudo-reflections.

Let

$$c : S \to \mathbb{C}$$

be a W-invariant function.

The rational Cherednik Algebra at $t = 0$ is

$$H_c := H_c(W, V) := T(V \oplus V^*) \rtimes W / \left\langle \begin{smallmatrix} 0=[x,x']=[y,y'] \text{ for } x,x' \in V^*, y,y' \in V \\ [y,x]=\sum_{s \in S} c_s \frac{\langle \alpha_s^\vee, x\rangle \langle y, \alpha_s\rangle}{\langle \alpha_s^\vee, \alpha_s\rangle} \cdot s \end{smallmatrix} \right\rangle$$

where $\langle -, - \rangle : V \times V^* \to \mathbb{C}$ is the standard pairing and $\alpha_s^\vee \in \mathrm{Im}(s-1) \subset V, \alpha_s \in \mathrm{Im}(s-1) \subset V^*$.

There is an isomorphism of vector spaces

$$H_c \simeq \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*].$$

Let Z_c be the center of H_c.

Basic facts:

1) Z_c is an integrally closed integral domain.

2)
$$Z_c \xrightarrow{\sim} eH_c e,$$
$$z \mapsto ze,$$

where
$$e = \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W.$$

3)
$$Z_c \supset \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W =: P,$$

and Z_c is free over P of rank $|W|$.

Example If $c = 0$ then $Z_0 = \mathbb{C}[V \times V^*]^W$.

Definition The **Calegero-Moser space** is

$$X_c := X_c(W, V) := \operatorname{Spec} Z_c,$$

a normal algebraic variety.

The map $\underline{\gamma} : X_c \to V/W \times V^*/W$ is finite and flat.

There is a grading on H_c, $\deg(x) = 1$, $\deg(y) = -1$, and $\deg(w) = 0$, which induces a $\mathbb{C}^\times$-action on X_c. There is an action of $\mathbb{C}^\times$ on $V/W \times V^*/W$ given by t acting by (t^{-1}, t). The map $\underline{\gamma}$ is $\mathbb{C}^\times$-equivariant, and $X_c^{\mathbb{C}^\times} = \underline{\gamma}^{-1}(0)$.

2 Representations of restricted rational Cherednik algebra

2.1

Let $\mathfrak{m}_P$ be the maximal ideal in P corresponding to $(0,0)$.

$$\overline{H}_c := H_c/\mathfrak{m}_p H_c \cong \mathbb{C}[V]^{\operatorname{Co} W} \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]^{\operatorname{Co} W}.$$

(Here $\operatorname{Co} W$ denotes coinvariants of W.)

For $\lambda \in \operatorname{Irr}(W)$, $\Delta_c(\lambda) := \operatorname{Ind}_{\mathbb{C}W \ltimes \mathbb{C}[V^*]^{\operatorname{Co} W}}^{\overline{H}_c}(\lambda)$ has unique simple quotient $L_c(\lambda)$.

Associated to λ we have $\chi_\lambda : Z_c \to \mathbb{C}$ the central character of $L(\lambda)$, this yields a map

$$z : \operatorname{Irr}(W) \to \underline{\gamma}^{-1}(0),$$

$$\lambda \mapsto \ker \chi_\lambda$$

surjective with fiber

$$\{\lambda \mid L(\lambda) \text{ same block}\}$$

which we call the **Calogero-Moser family**.

Theorem 2.1 *(Gordon, Bellamy-Schedler-Thiel)*

$$X_c \textit{ is smooth} \iff z \textit{ is bijective.}$$

2.2

We have a W-equivariant map

$$\Omega_c^H : H_c \to \mathbb{C},$$

$$pwq \mapsto p(0)wq(0),$$

using the triangular decomposition $H_c \simeq \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]$.

The restriction $\Omega_c = \Omega_c^H|_{Z_c}$ yields an algebra homomorphism on top here:

$$\begin{array}{ccc} Z_c & \xrightarrow{\Omega_c} & Z(\mathbb{C}W) \\ \downarrow & & \| \\ Z_c/\mathfrak{m}_p Z_c & & \| \\ \downarrow & & \| \\ \mathbb{C}[\underline{\gamma}^{-1}(0)] & \xhookrightarrow{z^*} & \bigoplus_{\chi \in \operatorname{Irr} W} \mathbb{C} \cdot e_\lambda \end{array}$$

Theorem 2.1 implies

$$X \text{ is smooth} \iff \Omega_c \text{ is surjective.}$$

For $w \in W$, set $a(w) = \operatorname{codim}(V^w)$.

$$F^i\mathbb{C}W := \operatorname{Span}\{w \mid a(w) \leq i\},$$

so

$$F^0\mathbb{C}W = \mathbb{C}1 \subset F^1\mathbb{C}W = \mathbb{C}\langle 1, s\rangle_{s \in S} \subset \cdots.$$

For $A \subset \mathbb{C}W$, we have $F^iA := A \cap F^i\mathbb{C}W$.

$$A \otimes \mathbb{C}[h] \supset \mathrm{Rees}(A) = \bigoplus_{i \geq 0} F^i A h^i$$

$$gr^{\bullet}(A) = \mathrm{Rees}^{\bullet}(A)|_{h=0}.$$

Conjecture 2.2 *(Bonnafé-Rouquier)*

a) $H^{2i+1}(X_c) = 0$ *for all* $i \geq 0$*,*

b) $H^*_{\mathbb{C}^\times}(X_c) \cong \mathrm{Rees}(\mathrm{Im}\,\Omega_c)$ *as* $H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[h]$*-algebras,*

c) $H^*(X_c) \cong \mathrm{gr}^{\bullet}(\mathrm{Im}\,\Omega_c)$*.*

Remark 1) If $c = 0$, $X_0 = V \times V^*/W$,

$$H^*(X_0) = H^0(X_0) = \mathbb{C} = \mathrm{Im}\,\Omega_0.$$

2) Assume a) and b) of the conjecture are true.

$$H^*_{\mathbb{C}^\times}(X_c) \otimes_{\mathbb{C}[h]} \mathbb{C}(h) \xrightarrow{\sim} H^*_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times}) \otimes_{\mathbb{C}[h]} \mathbb{C}(h) = \mathbb{C}(h)[\underline{\gamma}^{-1}(0)]$$
$$\cong \mathrm{Im}(\Omega_c) \otimes \mathbb{C}(h).$$

3) If X_c is smooth, then $\mathrm{Im}(\Omega_c) = Z(\mathbb{C}W)$, and a), c) of conjecture have been proved by Etingof-Ginzburg.

4) If W is cyclic then the conjecture is true.

Theorem 2.3 *(Bonnafé-Shan)*

1) If X_c *is smooth, then part b) holds.*

2) Assume X_0 *has a symplectic resolution*

$$\mathcal{X} \xrightarrow{\pi} X_0.$$

Then the $\mathbb{C}^\times$*-action lifts to* $\mathcal{X}$*.*

$$H^*_{\mathbb{C}^\times}(\mathcal{X}) \cong \mathrm{Rees}(Z(\mathbb{C}W)).$$

We describe the proof of 1).

3 Idea of proof

1) From Etingof and Ginzburg we already know $H^{\mathrm{odd}}(X_c) = 0$, and X_c is equivariantly formal.

Let

$$i : X_c^{\mathbb{C}^\times} \hookrightarrow X_c,$$

$$i^* : H^*_{\mathbb{C}^\times}(X_c \hookrightarrow H^*_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times})$$

where

$$H^*_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times}) = \bigoplus_{\lambda \in \mathrm{Irr}(W)} \mathbb{C}[h] \cdot z(\lambda) = \mathbb{C}[h] \otimes Z(\mathbb{C}W)$$

$$z(\lambda) \mapsto e_\lambda,$$

where e_λ is the central primitive idempotent in $\mathbb{Z}(\mathbb{C}W)$ corresponding to the irreducible representation λ.

Problem: want to show

$$\mathrm{Im}(i^*) = \mathrm{Rees}(Z(\mathbb{C}W)),$$

enough to show

$$\mathrm{Im}(i^*) \supseteq \mathrm{Rees}(Z(\mathbb{C}W)).$$

Recall

$$\mathrm{Rees}(Z(\mathbb{C}W)) = \bigoplus_{r\geq 0} F^r(Z(\mathbb{C}W))h^r.$$

Let

$$P^r(W) := \{W' \subset W \text{ parabolic subgroup} \mid \mathrm{codim}(V^{W'}) = r\}.$$

$$\begin{aligned} \mathrm{Tr} : Z(\mathbb{C}W') &\to Z(\mathbb{C}W) \\ z &\mapsto \sum_{g\in W/W'} gzg^{-1}. \end{aligned}$$

$$\mathcal{F}^r(Z(\mathbb{C}W)) = \sum_{W'\in\mathcal{P}^r(W)} \mathrm{Tr}(Z(\mathbb{C}W')).$$

Problem: For $\chi' \in \mathrm{Irr}(W')$,

$$\mathrm{Tr}(e^{W'}_{\chi'})h^r \in \mathrm{Im}(i^*).$$

Consider

$$K_{\mathbb{C}^\times}(X_c) = \text{Grothendieck group of } \mathbb{Z}\text{-graded projective } Z_c\text{-modules}.$$

$$\begin{array}{ccccc} & & K_{\mathbb{C}^\times}(X_c) & & \\ & \swarrow^{Ch} & & \searrow^{i^*} & \\ \widehat{H^*_{\mathbb{C}^\times}}(X_c) & & & & K_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times}) \\ & \searrow_{i^*} & & \swarrow_{Ch} & \\ & & \widehat{H^*_{\mathbb{C}^\times}}(X_c^{\mathbb{C}^\times}) & & \end{array}$$

Here

$$K_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times}) = \bigoplus_{\lambda\in\mathrm{Irr}(W)} \mathbb{C}[q^{\pm 1}]z(\lambda)$$

and $i^* : K_{\mathbb{C}^\times}(X_c) \to K_{\mathbb{C}^\times}(X_c^{\mathbb{C}^\times})$ is given by

$$P \mapsto \bigoplus_{\lambda\in\mathrm{Irr}(W)} \mathrm{gdim}(P/\mathfrak{m}_\lambda).$$

For $E \in \mathrm{Irr}(W)$, $P(E) = H_c \otimes_{\mathbb{C}W} E \in \mathrm{Proj}(H_c)$.

In the smooth case we have a Morita equivalence

$$\begin{aligned} Z_c\text{-mod} &\overset{\sim}{\leftarrow} H_c\text{-mod} \\ e(M) &\longleftarrow\!\shortmid M. \end{aligned}$$

We have $eP(E) \in \mathrm{Proj}(Z_c)$.

We can compute $Ch(eP(E))$, $Ch(eP(\mathrm{Ind}^W_{W'}(\Lambda^*(V^{W'})^\perp \otimes \chi')))$.