Period Polynomial Relations among Double Zeta Values and Various Generalizations

MA, Ding

Duke University

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What is the Riemann zeta function?

Definition (Riemann zeta function)

The Riemann zeta function, $\zeta(s)$, is a function of a complex variable s that analytically continues the sum of infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges only when the $\mathrm{Re}(s) > 1$.

Special values of $\zeta(s)$

- For any positive even integer $2n$:

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

The numerator of Bernoulli numbers may contain a special kind of prime numbers, called irregular primes. The first example of irregular prime is 691, which appears in

$$B_{12} = -\frac{691}{2730}.$$

What are multiple zeta values?

Now we restrict ourselves to the values of the Riemann zeta function at positive integers.

Definition (zeta values)

Any $\zeta(n)$ for an integer $n \geq 2$ is called a zeta value.

Definition (multiple zeta values)

A multiple zeta value (MZV) is a real number of the form

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},$$

where $n_1, \ldots, n_{r-1} \geq 1$, $n_r \geq 2$ are integers. The number r is called the depth, and the number $N := n_1 + \cdots + n_r$ is called the weight.

These numbers were first defined by Euler for $r = 2$ in late 1700's, and they were popularized by Zagier in the 1990's, who discovered that they satisfy vast numbers of relations.

MZVs are iterated integrals

Definition (iterated integral)

Let M be a differentiable manifold and let $\omega_1, \ldots, \omega_n$ be smooth 1-forms on M. Consider a smooth path $\gamma : (0,1) \to M$. The iterated integral of $\omega_1 \cdots \omega_n$ along γ is defined (when it converges) by

$$\int_\gamma \omega_1 \cdots \omega_n := \int_{0<t_1<\cdots<t_n<1} \gamma^*(\omega_1)(t_1) \cdots \gamma^*(\omega_n)(t_n).$$

Kontsevich showed that when $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\gamma(t) = t$ is simply the inclusion $(0,1) \to M$, one has the following iterated integral representation of MZVs

$$\zeta(n_1, \ldots, n_r) = \int_\gamma \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n_1 - 1} \cdots \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n_r - 1},$$

where $\omega_0 = \frac{dt}{t}$ and $\omega_1 = \frac{dt}{1-t}$.
The shuffle product formula of iterated integral guarantees that the product of MZVs is a linear combination of MZVs.

motivic MZVs

Since the standard transcendence conjectures for MZVs are inaccessible, one can replace the study of MZVs with motivic MZVs

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r),$$

which are elements of a certain algebra $\mathcal{H} = \bigoplus_k \mathcal{H}_k$ over $\mathbb{Q}$, which is graded by weight. (Again, the shuffle product formula of motivic iterated integral guarantees that the product of motivic MZVs is a linear combination of motivic MZVs.)
There is a period map

$$\begin{array}{rcl} \mathrm{per}: \quad \mathcal{H} & \to & \mathbb{R} \\ \zeta^{\mathfrak{m}}(n_1, \dots, n_r) & \mapsto & \zeta(n_1, \dots, n_r). \end{array}$$

Since the depth filtration $\mathfrak{D}$ is motivic, one can define its associated graded algebra $\mathrm{gr}^{\mathfrak{D}}\mathcal{H}$. The depth-graded motivic MZV

$$\zeta^{\mathfrak{m}}_{\mathfrak{D}}(n_1, \dots, n_r) \in \mathrm{gr}^{\mathfrak{D}}_r \mathcal{H}$$

is given by the class of $\zeta^{\mathfrak{m}}(n_1, \dots, n_r)$ modulo elements of lower depth.

Parity result

Theorem (Parity result)

When $k \not\equiv r \pmod 2$, we have

$$\zeta^{\mathfrak{m}}_{\mathfrak{D}}(n_1, \ldots, n_r) \equiv 0 \pmod{\mathrm{gr}^{\mathfrak{D}}_{r-1}\mathcal{H} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]}.$$

Example

When $r = 2$ and k is odd, we have

- $\zeta^{\mathfrak{m}}(1,2) = \zeta^{\mathfrak{m}}(3)$
- $\zeta^{\mathfrak{m}}(1,4) = -\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) + 2\zeta^{\mathfrak{m}}(5)$
- $\zeta^{\mathfrak{m}}(2,3) = 3\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(2) + \frac{11}{2}\zeta^{\mathfrak{m}}(5)$

  $\vdots$

- $\zeta^{\mathfrak{m}}(2,9) = 9\zeta^{\mathfrak{m}}(9)\zeta^{\mathfrak{m}}(2) + 6\zeta^{\mathfrak{m}}(7)\zeta^{\mathfrak{m}}(4) + 4\zeta^{\mathfrak{m}}(5)\zeta^{\mathfrak{m}}(6) + 2\zeta^{\mathfrak{m}}(3)\zeta^{\mathfrak{m}}(8) + 28\zeta^{\mathfrak{m}}(11)$

  $\vdots$

But how can we compute those coefficients?

Decomposition formula for double zeta values of odd weight

In 2012, Zagier proved the following decomposition formula for double zeta values of odd weight, the existence of which was first predicted by Euler without the explicit formula in 1770's.

Theorem (Zagier, 2012)

The double zeta value $\zeta(m, n)$ $(m \geq 1, n \geq 2)$ of odd weight $m + n = k$ satisfies

$$\zeta(m,n) = (-1)^m \sum_{s=0}^{\frac{k-3}{2}} \Bigg[\underbrace{\binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n,2s} + (-1)^m \delta_{s,0}}_{\text{an integer}} \Bigg] \zeta(2s)\zeta(k-2s),$$

where $\zeta(0) := -\frac{1}{2}$ by convention. Indeed, the same formula works for the motivic MZVs.

What about double zeta values of even weight?

In 2006, Gangl, Kaneko and Zagier found a family of relations coming from cusp forms of level 1 among double zeta values

$$\{\zeta(3,k-3),\zeta(5,k-5),\dots,\zeta(k-3,3)\} \longleftarrow \text{totally odd}$$

of weight k modulo $\mathbb{Q}\zeta(k)$, when k is even.

The number of such relations (linearly independent ones) is exactly the dimension of cusp form of weight k. Again, such relations are also relations among the motivic double zeta values.

The first of such example happens in weight 12:

$$\begin{aligned} 14\zeta(3,9)+75\zeta(5,7)+84\zeta(7,5) &= \frac{5197}{691}\zeta(12) \\ &\equiv 0 \pmod{\mathbb{Q}\zeta(12)}. \end{aligned}$$

Broadhurst-Kreimer conjecture

Conjecture (motivic Broadhurst-Kreimer conjecture)

The generating series of the dimensions of the spaces $\mathrm{gr}_r^{\mathfrak{D}}\mathcal{H}_k$ *is given by*

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_r^{\mathfrak{D}}\mathcal{H}_k)x^k y^r = \frac{1+\mathbb{E}(x)y}{1-\mathbb{O}(x)y+\mathbb{S}(x)(y^2-y^4)},$$

where

$$\mathbb{E}(x) = \frac{x^2}{1-x^2}, \quad \mathbb{O}(x) = \frac{x^3}{1-x^2}, \quad \mathbb{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)}.$$

Example

$$r=1: \quad \sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_1^{\mathfrak{D}}\mathcal{H}_k)x^k = \mathbb{O}(x)+\mathbb{E}(x)$$

$$r=2: \quad \sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_2^{\mathfrak{D}}\mathcal{H}_k)x^k = \mathbb{O}^2(x)-\mathbb{S}(x)+\mathbb{E}(x)\mathbb{O}(x)$$

Summary

1. Parity result

2. Broadhurst-Kreimer conjecture

$$\sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_2^{\mathfrak{D}}\mathcal{H}_k)x^k = \underbrace{\mathbb{O}^2(x) - \mathbb{S}(x)}_{k \text{ even}} + \underbrace{\mathbb{E}(x)\mathbb{O}(x)}_{k \text{ odd}}.$$

   This $\mathbb{S}(x)$ is given by those GKZ-type relations.

3. Euler-Zagier decomposition formula of double zeta values of odd weight

Question: What about triple zeta values of even weight?

4. Gangl-Kaneko-Zagier type relations of double zeta values of even weight

Question: Any other GKZ-type relations?

Topics

In this talk, we will cover the following results:

1. GKZ-type relations (arising from cusp forms) among double zeta values of odd weight
2. generalization of the Euler-Zagier decomposition formula to triple zeta values of even weight
3. generalization of Ihara-Takao's result to double zeta values of level 2 and 3
4. generalization of Eichler-Shimura-Manin correspondence to newforms of level 2 and 3

GKZ-type relation among double zeta values of even weight

The period polynomial of a cusp form of weight k and level 1 is defined to be

$$r_f(X,Y) = \int_0^{i\infty} f(\tau)(\tau Y - X)^{k-2} d\tau.$$

Theorem (Gangl-Kaneko-Zagier, 2006)

Let $k \geq 12$ be an even integer, let

$$P(X,Y) = \sum_{i=1}^{\lfloor \frac{k-4}{4} \rfloor} a_i (X^{2i} Y^{k-2-2i} - Y^{2i} X^{k-2-2i})$$

be a restricted even period polynomial of weight k, and write

$$P(X+Y,Y) = \sum_{r=1}^{k-3} \binom{k-2}{r-1} q_{r,k-r} X^{r-1} Y^{k-r-1}.$$

Then the linear combination satisfies

$$\sum_{r=3:\ \mathrm{odd}}^{k-3} q_{r,k-r} \zeta(k-r,r) \equiv 0 \pmod{\mathbb{Q}\zeta(k)}.$$

Moreover, every linear relation arises in this way.

Examples of GKZ-type relation

The even period polynomial (up to a scalar) of Δ (weight 12 cusp form) is

$$\frac{36}{691}(X^{10}-Y^{10})-(X^8Y^2-3X^6Y^4+3X^4Y^6-X^2Y^8).$$

The restricted even period polynomial is

$$\begin{aligned} P(X,Y) &= X^8Y^2-3X^6Y^4+3X^4Y^6-X^2Y^8 \\ P(X+Y,Y) &= 1X^8Y^2+8X^7Y^3+25X^6Y^4+38X^5Y^5+28X^4Y^6+8X^3Y^7 \end{aligned}$$

From

$$\begin{aligned} 630q_{9,3} &= 630\frac{1}{\binom{10}{8}}=14 \\ 630q_{7,5} &= 630\frac{25}{\binom{10}{6}}=75 \\ 630q_{5,7} &= 630\frac{28}{\binom{10}{4}}=84, \end{aligned}$$

Gangl-Kaneko-Zagier's result tells us

$$14\zeta(3,9)+75\zeta(5,7)+84\zeta(7,5)\equiv 0 \pmod{\mathbb{Q}\zeta(12)}.$$

Brown's isomorphism

In order to generalize GKZ's result to the motivic MZVs, we need the following result.

Theorem (Brown, 2013)

There exists a non-canonical algebra isomorphism

$$\begin{aligned} \phi: \quad \mathcal{H} &\to \mathbb{Q}\langle f_3, f_5, \ldots, f_{2n+1}, \ldots\rangle \otimes_{\mathbb{Q}} \mathbb{Q}[f_2] \\ \zeta^{\mathfrak{m}}(k) &\mapsto f_k, \end{aligned}$$

where $f_{2n} := b_n f_2^n$ *with* $b_n \in \mathbb{Q}^\times$ *satisfying* $\zeta(2n) = b_n \zeta(2)^n$. *It preserves the weight grading and the depth filtration. The product in* $\mathbb{Q}\langle f_3, f_5, \ldots, f_{2n+1}, \ldots\rangle$ *is given by the shuffle product.*

Example

When r and s are both odd, we have

$$\phi(\zeta^{\mathfrak{m}}(r,s) + \zeta^{\mathfrak{m}}(s,r)) = \phi(\zeta^{\mathfrak{m}}(r)\zeta^{\mathfrak{m}}(s) - \zeta^{\mathfrak{m}}(k)) = f_r ш f_s - f_k = f_r f_s + f_s f_r - f_k.$$

When r is odd and $s = 2$, we have

$$\phi(\zeta^{\mathfrak{m}}(r,2) + \zeta^{\mathfrak{m}}(2,r)) = \phi(\zeta^{\mathfrak{m}}(r)\zeta^{\mathfrak{m}}(2) - \zeta^{\mathfrak{m}}(k)) = f_r f_2 - f_k.$$

Reformulations of GKZ's result in the motivic setting

When $k \geq 6$ is an even integer, there is a matrix $\mathcal{A}_{2,k,\mathbf{oo}}$, firstly defined by Baumard-Schneps, and later by Brown in the motivic setting, such that

$$(f_{k-3}f_3, f_{k-5}f_5, \dots, f_3 f_{k-3}) \cdot \mathcal{A}_{2,k,\mathbf{oo}} \equiv (\phi(\zeta^{\mathfrak{m}}(k-3,3)), \phi(\zeta^{\mathfrak{m}}(k-5,5)), \cdots, \phi(\zeta^{\mathfrak{m}}(3,k-3))).$$

Example

For example, when $k = 12$, we have

$$\mathcal{A}_{2,12,\mathbf{oo}} = \begin{pmatrix} 28 & 42 & -42 & -27 \\ 15 & 15 & -14 & -15 \\ 6 & 1 & 0 & -6 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It satisfies that

$$(f_9 f_3, f_7 f_5, f_5 f_7, f_3 f_9) \cdot \mathcal{A}_{2,12,\mathbf{oo}} \equiv (\phi(\zeta^{\mathfrak{m}}(9,3)), \phi(\zeta^{\mathfrak{m}}(7,5)), \phi(\zeta^{\mathfrak{m}}(5,7)), \phi(\zeta^{\mathfrak{m}}(3,9))).$$

GKZ's result can be reformulated as studying the right annihilator of $\mathcal{A}_{2,k,\mathbf{oo}}$. The only element $(0, 84, 75, 14)^T$ (up to a scalar) in the right annihilator gives

$$84\zeta^{\mathfrak{m}}(7,5) + 75\zeta^{\mathfrak{m}}(5,7) + 14\zeta^{\mathfrak{m}}(3,9) \equiv 0 \pmod{\mathbb{Q}\zeta^{\mathfrak{m}}(12)}.$$

Other known results about $\mathcal{A}_{2,k,\mathbf{oo}}$

- The left annihilator of $\mathcal{A}_{2,k,\mathbf{oo}}$ is also well-understood by the previous work of Ihara-Takao, Goncharov, Baumard-Schneps, Hain-Matsumoto, Brown, and etc. The left annihilator consists of exactly the coefficients of the restricted even period polynomials of the cusp forms of weight k.

Example

For example, when $k = 12$, we have

$$\mathcal{A}_{2,12,\mathbf{oo}} = \begin{pmatrix} 28 & 42 & -42 & -27 \\ 15 & 15 & -14 & -15 \\ 6 & 1 & 0 & -6 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\ker \mathcal{A}_{2,12,\mathbf{oo}} = \langle (1, -3, 3, -1) \rangle_{\mathbb{Q}}.$$

- When k is even, $\mathcal{A}_{2,k,\mathbf{ee}}$ has been completely studied by Kaneko and Tasaka.

Euler-Zagier's decomposition formula revisited

Amazingly, Euler-Zagier's decomposition formula can also be rewritten in such a matrix form.

Theorem (Zagier (motivic version))

The motivic double zeta value $\zeta^{\mathfrak{m}}(m,n)$ $(m \geq 1, n \geq 2)$ *of odd weight* $m+n=k$ *satisfies*

$$\zeta^{\mathfrak{m}}(m,n) = (-1)^m \sum_{s=0}^{\frac{k-3}{2}} \left[\binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n,2s} + (-1)^m \delta_{s,0} \right] \zeta^{\mathfrak{m}}(2s)\zeta^{\mathfrak{m}}(k-2s).$$

Remember that

$$\phi(\zeta^{\mathfrak{m}}(2i+1)\zeta^{\mathfrak{m}}(2j)) = f_{2i+1}f_{2j},$$

we can reformulate Zagier's decomposition formula by using the following two matrices

$$\mathcal{A}_{2,k,\mathbf{eo}} \text{ and } \mathcal{A}_{2,k,\mathbf{oe}}.$$

Remark

Those four matrices $\mathcal{A}_{2,k,\mathbf{oo}}$, $\mathcal{A}_{2,k,\mathbf{ee}}$, $\mathcal{A}_{2,k,\mathbf{eo}}$, and $\mathcal{A}_{2,k,\mathbf{oe}}$ have a unified expression.

Zagier's results and questions

Now when k is odd, we want to describe the left annihilator and the right annihilator of $\mathcal{A}_{2,k,\mathbf{eo}}$, and $\mathcal{A}_{2,k,\mathbf{oe}}$.

Theorem (Zagier, 2012)

For any odd $k \geq 5$, the determinant of the matrix $\mathcal{A}_{2,k,\mathbf{eo}}$ is nonzero.

Theorem (Zagier, 2012)

For any odd $k \geq 11$, there is an injection $i : \mathcal{S}_{k-1} \oplus \mathcal{S}_{k+1} \to \ker \mathcal{A}_{2,k,\mathbf{oe}}$, where $i|_{\mathcal{S}_{k-1}}$ is given by the coefficients of the odd period polynomials, and $i|_{\mathcal{S}_{k+1}}$ is given by the coefficients of the partial derivative $\frac{\partial}{\partial X}$ of the restricted even period polynomials.

Question (Zagier)

How to characterize the right annihilator of $\mathcal{A}_{2,k,\mathbf{oe}}$?
($\Longleftrightarrow$ describe the linear relations among $\{\zeta^{\mathfrak{m}}(odd, even)\}$)

Recall GKZ's result

GKZ's result generates the period polynomial relations between double zeta values of even weights.

Theorem (Gangl-Kaneko-Zagier, 2006)

Let $k \geq 12$ be an even integer, let

$$P(X,Y) = \sum_{i=1}^{\lfloor \frac{k-4}{4} \rfloor} a_i (X^{2i} Y^{k-2-2i} - Y^{2i} X^{k-2-2i})$$

be a restricted even period polynomial of weight k, and write

$$P(X+Y,Y) = \sum_{r=1}^{k-3} \binom{k-2}{r-1} q_{r,k-r} X^{r-1} Y^{k-r-1}.$$

Then the linear combination satisfies

$$\sum_{r=3:\ \mathrm{odd}}^{k-3} q_{r,k-r} \zeta(k-r,r) \equiv 0 \pmod{\mathbb{Q}\zeta(k)}.$$

Moreover, every linear relation arises in this way.

Answers to Zagier's question

My result generates the period polynomial relations between double zeta values of odd weights, and there are two families of them.

Theorem (Ma, 2016)

(Type I): For each odd period polynomial p of weight k, we define $b_{r,s}$ by

$$p(X+Y,Y) = \sum_{r+s=k+1} \binom{k-1}{r-1} b_{r,s} X^{r-1} Y^{s-2}.$$

Then

$$\sum_{\substack{r+s=k+1 \\ 4\le r\le k-2:\ \text{even}}} (b_{r,s} - b_{s,r})\zeta(s,r) \equiv 0 \pmod{\mathbb{Q}\zeta(k+1)}.$$

(Type II): For each restricted even period polynomial p of weight k, we define $c_{r,s}$ by

$$\frac{\partial}{\partial X} p(X+Y,Y) = \sum_{r+s=k-1} \binom{k-3}{r-1} c_{r,s} X^{r-1} Y^{s-1}.$$

Then

$$\sum_{\substack{r+s=k-1 \\ 4\le r\le k-4:\ \text{even}}} (c_{r,s} - c_{s,r})\zeta(s,r) \equiv 0 \pmod{\mathbb{Q}\zeta(k-1)}.$$

Two examples

The weight 12 cusp form produces two relations. One in weight 11, and the other in weight 13.

Example

- The weight 13 relation coming from odd period polynomial of weight 12 is

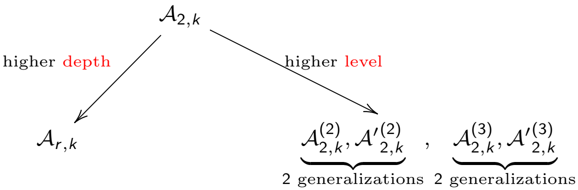
$$\begin{aligned} P(X,Y) &= 4X^9Y^1 - 25X^7Y^3 + 42X^5Y^5 - 25X^3Y^7 + 4X^1Y^9 \\ -3\zeta(13) &= 24\zeta(3,10) + 28\zeta(5,8) - 10\zeta(7,6) - 36\zeta(9,4) \end{aligned}$$

- The weight 11 relation coming from restricted even period polynomial of weight 12 is

$$\begin{aligned} P(X,Y) &= X^8Y^2 - 3X^6Y^4 + X^4Y^6 - X^2Y^8 \\ -3\zeta(11) &= 28\zeta(3,8) + 20\zeta(5,6) - 42\zeta(7,4) \end{aligned}$$

Two generalizations of $\mathcal{A}_{2,k}$

There are two possible generalizations of the matrices $\mathcal{A}_{2,k}$ that we discussed before.



- The matrices $\mathcal{A}_{r,k}$ was first defined by Brown for the case of totally odd indexing set $\mathbf{oo}\cdots\mathbf{o}$. In such case, Koji Tasaka gave a closed formula in his thesis. Later, Tasaka and I generalized such a formula to arbitrary indexing set. In particular, in depth 2, our formula unified $\mathcal{A}_{2,k,\mathbf{oo}}$, $\mathcal{A}_{2,k,\mathbf{ee}}$, $\mathcal{A}_{2,k,\mathbf{eo}}$, and $\mathcal{A}_{2,k,\mathbf{oe}}$.

Two generalizations of $\mathcal{A}_{2,k}$

- But for $r \geq 4$, the right annihilator of $\mathcal{A}_{r,k}$ may not always give us linear relations between MZVs.

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_r^{\mathfrak{D}} \mathcal{H}_k) x^k y^r = \frac{1+\mathbb{E}(x)y}{1-\mathbb{O}(x)y+\mathbb{S}(x)(y^2-y^4)},$$

$$r=4: \quad \sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathrm{gr}_4^{\mathfrak{D}} \mathcal{H}_k) x^k = \mathbb{O}^4(x) - 3\mathbb{O}(x)\mathbb{S}(x) + \underbrace{\mathbb{S}(x)}_{\text{fake relations}} + \cdots$$

- The matrices $\mathcal{A}_{2,k}^{(2)}$, $\mathcal{A'}_{2,k}^{(2)}$, $\mathcal{A}_{2,k}^{(3)}$, $\mathcal{A'}_{2,k}^{(3)}$ were defined by me using results of Deligne, Goncharov, and Glanois.

Decomposition formulas for triple zeta values

One result we obtained by studying $\mathcal{A}_{3,k}$ (k even) is the following decomposition formula.

Theorem (Ma-Tasaka, 2016)

For any even integer $k \geq 8$, we have

$$\zeta^{\mathfrak{m}}(n_1, n_2, n_3) \equiv \sum_{\substack{k_1, k_2 \geq 2 \\ k_3 \geq 2:\ \text{even}}} n\binom{k_1, k_2, k_3}{n_1, n_2, n_3} \zeta^{\mathfrak{m}}(k_1, k_2) \zeta^{\mathfrak{m}}(k_3) \pmod{\textit{lower depth terms}},$$

with explicit formulas for $n\binom{k_1, k_2, k_3}{n_1, n_2, n_3}$.

Such a result generalizes Zagier's decomposition formula for the double zeta values of odd weight in the motivic setting.
I want to mention that Erik Panzer also proved such a decomposition formula for the classic triple zeta values of even weights.

MZVs of level N

Multiple zeta values of level N are defined by

$$\zeta\binom{n_1,\cdots,n_r}{\varepsilon_1,\cdots,\varepsilon_r} := \sum_{0<k_1<\cdots<k_r} \frac{\varepsilon_1^{k_1}\cdots\varepsilon_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}, \quad \varepsilon_i \in \mu_N,\ (n_r,\varepsilon_r) \neq (1,1).$$

We also have the corresponding motivic version for them, denoted by $\zeta^{\mathfrak{m}}(\cdot)$.

Theorem (Deligne, Deligne-Goncharov)

- *For $N = 2$,*

$$(2^{-2n}-1)\zeta^{\mathfrak{m}}\binom{2n+1}{1} \equiv \zeta^{\mathfrak{m}}\binom{2n+1}{-1}.$$

- *For $N = 3$,*

$$\zeta^{\mathfrak{m}}\binom{2n+1}{1}(1-3^{2n}) \equiv 2\cdot 3^{2n}\zeta^{\mathfrak{m}}\binom{2n+1}{\varepsilon_3}, \quad \zeta^{\mathfrak{m}}\binom{2n}{1} \equiv 0, \quad \zeta^{\mathfrak{m}}\binom{n}{\varepsilon_3} \equiv (-1)^{n-1}\zeta^{\mathfrak{m}}\binom{n}{\varepsilon_3^{-1}},$$

where $\varepsilon_3 = e^{\frac{2\pi\sqrt{-1}}{3}}$.

Why two generalizations $\mathcal{A}_{2,k}^{(N)}$ and $\mathcal{A'}_{2,k}^{(N)}$?

Recall that

$$(f_{k-3}f_3, f_{k-5}f_5, \ldots, f_3 f_{k-3}) \cdot \mathcal{A}_{2,k,\mathbf{oo}} \equiv (\phi(\zeta^{\mathfrak{m}}(k-3,3)), \phi(\zeta^{\mathfrak{m}}(k-5,5)), \cdots, \phi(\zeta^{\mathfrak{m}}(3,k-3))).$$

There is only one choice for each $f_{2n+1} = \phi(\zeta^{\mathfrak{m}}(2n+1))$.
But when $N = 2, 3$, we can choose

$$f_{2n+1} = \phi_N\left(\zeta^{\mathfrak{m}}\binom{2n+1}{1}\right) \quad \text{or} \quad f'_{2n+1} = \phi_N\left(\zeta^{\mathfrak{m}}\binom{2n+1}{\varepsilon_N}\right).$$

(They only differ by a scalar.)
Roughly speaking, two generalizations $\mathcal{A}_{2,k}^{(N)}$ and $\mathcal{A'}_{2,k}^{(N)}$ correspond to

$$(f_{k-3}f_3, f_{k-5}f_5, \ldots, f_3 f_{k-3}) \cdot \mathcal{A}_{2,k}^{(N)} \equiv \left(\phi_N(\zeta^{\mathfrak{m}}\tbinom{k-3,3}{\varepsilon_N,\varepsilon_N^{-1}}), \phi_N(\zeta^{\mathfrak{m}}\tbinom{k-5,5}{\varepsilon_N,\varepsilon_N^{-1}}), \cdots, \phi_N(\zeta^{\mathfrak{m}}\tbinom{3,k-3}{\varepsilon_N,\varepsilon_N^{-1}})\right).$$

$$(f'_{k-3}f_3, f'_{k-5}f_5, \ldots, f'_3 f_{k-3}) \cdot \mathcal{A'}_{2,k}^{(N)} \equiv \left(\phi_N(\zeta^{\mathfrak{m}}\tbinom{k-3,3}{\varepsilon_N,\varepsilon_N^{-1}}), \phi_N(\zeta^{\mathfrak{m}}\tbinom{k-5,5}{\varepsilon_N,\varepsilon_N^{-1}}), \cdots, \phi_N(\zeta^{\mathfrak{m}}\tbinom{3,k-3}{\varepsilon_N,\varepsilon_N^{-1}})\right).$$

Left eigenvectors of $\mathcal{A}_{2,k}^{(N)}$

The left eigenvectors of $\mathcal{A}_{2,k}^{(N)}$ are related to the level 1 Hecke eigenforms and Hecke eigenvalues.

Theorem (Ma, 2016)

Let k be an even integer. When $N = 2, 3$, the vectors coming from the restricted even period polynomials of cuspidal eigenforms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ are left eigenvectors of $\mathcal{A}_{2,k}^{(N)}$, and the corresponding eigenvalues are given by

- $N = 2$,

$$\frac{\lambda_2 - (1 + 2^{k-1})}{2^{k-2}},$$

- $N = 3$,

$$\frac{\lambda_3 - (1 + 3^{k-1})}{4 \cdot 3^{k-2}},$$

where λ_2 (respectively, λ_3) is the eigenvalue of the Hecke operator T_2 (respectively, T_3) for the corresponding eigenform.

Left eigenvectors of $\mathcal{A}'^{(N)}_{2,k}$

The left eigenvectors of $\mathcal{A}'^{(N)}_{2,k}$ are related to the newforms of level $\Gamma_1(N)$.

Theorem (Ma, 2016)

Let k be an even integer. When $N = 2, 3$, the vectors coming from the restricted even period polynomials of newforms of weight k and level $\Gamma_1(N)$ are left eigenvectors of $\mathcal{A}'^{(N)}_{2,k}$, and the corresponding eigenvalues are given by

- $N = 2$,

$$-\left(1 + \frac{\varepsilon}{2^{\frac{k-2}{2}}}\right),$$

- $N = 3$,

$$-\frac{1}{2}\left(1 + \frac{\varepsilon}{3^{\frac{k-2}{2}}}\right),$$

where $\varepsilon \in \{\pm 1\}$ is the eigenvalue of the Atkin-Lehner involution W_N on the corresponding newform.

Why do both of them generalize the classic result

It is worth mentioning that in the proofs, we can see that the actions of $\mathcal{A}_{2,k}^{(N)}$ and $\mathcal{A}'^{(N)}_{2,k}$ are nothing but the following two well-known operators (up to scalar) on the corresponding spaces:

$$\mathcal{A}_{2,k}^{(N)} \longleftrightarrow T_N - 1 - N^{k-1} \quad \text{acting on } r_f^{-,0}(x,y) \text{ of } f \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$$
$$\mathcal{A}'^{(N)}_{2,k} \longleftrightarrow U_N - 1 \quad \text{acting on } r_f^{-,0}(x,y) \text{ of } f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_1(N))^{\pm},$$

where $r_f^{-,0}(x,y)$ denotes the restricted even period polynomial of f.

Note that those two theorems are compatible with the classic result, since when $N=1$ both $\mathcal{A}_{2,k}^{(N)}$ and $\mathcal{A}'^{(N)}_{2,k}$ give us the

$$T_1 - 1 = U_1 - 1 = 0$$

action on the restricted even period polynomials of $f \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ (in this case, $\mathcal{S}_k^{\mathrm{new}}(\mathrm{SL}_2(\mathbb{Z})) = \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$).

Eichler-Shimura-Manin correspondence for newforms of level 2 and level 3

Also the matrices $\mathcal{A}'^{(N)}_{2,k}$ give us a characterization of the restricted even period polynomial of a newform of level 2 and 3.

Theorem (level 1 Eichler-Shimura-Manin correspondence)

The map

$$r^{-,0} : \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z})) \to \mathbf{W}_k^{-,0}$$

is an isomorphism over $\mathbb{C}$

Conjecture (Ma, 2016)

We have the following isomorphisms defined over $\mathbb{C}$*:*

$$\mathcal{S}_k^{\mathrm{new}}(\Gamma_1(2))^{\pm} \cong (\mathbf{W}_{k,\mathrm{new}}^{(2),-,0})^{\pm} := \left\{ p(x,y) \in \mathbb{C}[x,y] \;\middle|\; \begin{array}{l} -p(y,x)-p(y,x+y) \\ \quad +p(x,x+y)=-p(x,y) \\ -p(y,2x)=\pm 2^{\frac{k-2}{2}} p(x,y) \end{array} \right\},$$

$$\mathcal{S}_k^{\mathrm{new}}(\Gamma_1(3))^{\pm} \cong (\mathbf{W}_{k,\mathrm{new}}^{(3),-,0})^{\pm} := \left\{ p(x,y) \in \mathbb{C}[x,y] \;\middle|\; \begin{array}{l} -p(y,x)-p(y,x+y)+p(x,x+y) \\ \quad -p(y,x-y)+p(x,x-y)=-p(x,y) \\ -p(y,3x)=\pm 3^{\frac{k-2}{2}} p(x,y) \end{array} \right\}.$$

THE END

Hope you enjoy!