This report contains observations made after the conference. Y. André mentioned [6] and L. Ma sent [4].

There are several versions of the definition of big Cohen-Macaulay modules. Let R be a noetherian local ring of dimension d. We say that an R-module M is CM iff depth $M = d$ (depth defined by local cohomology). One says that M is CM w.r.t. a system of parameters $(x_1, \ldots, x_d) = \underline{x}$ iff $\underline{x}$ is a regular sequence on M (in the sense of EGA, i.e. without a non-triviality assumption) and $M \neq \underline{x}M$. One says that M is balanced big CM if it is CM w.r.t. any s.o.p. of R.
For a noetherian ring R we say that an R-module M is secantly CM iff every strictly secant sequence $x_1, \ldots, x_d$ of R is M-regular. This is equivalent to the condition that $\operatorname{depth}_{\mathfrak{p}}(M_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$ or ∞ $\forall\, \mathfrak{p} \in \operatorname{Spec}(R)$. [There is an almost variant.]
One says that an R-module M is valuctively non trivial (vnt) if for every homomorphism $R \to W$ to a valuation ring W with residue field k, $((M \otimes_R W)/\text{torsion}) \otimes_W k \neq 0$.

For M an R-algebra B this is equivalent to saying that $\operatorname{Spec}(B) \to \operatorname{Spec}(R)$ is covering for the v-topology, or that $R \to B$ is strongly submersive in the sense of Nagata.

If a module M over a noetherian domain is solid in the sense of [4] then it is vnt. Any CM module over a complete local noetherian domain is solid.

Over a noetherian local R, if M is CM, $\hat{M}$ is balanced CM.
If R is catenerian and equidimensional, secantly CM is the same as balanced CM without the non triviality condition.

If M is secantly CM and vnt over R then R is locally equidimensional and universally catenarian.

<u>Claim</u>. If R is noetherian, x-adically complete (separated), x non zero divisor on R and on a module M, and M/xM is secantly CM over R/x, then $\hat{M}$ is secantly CM over R. [There is an almost variant.]

If $R \xrightarrow{\varphi} S$ is a homomorphism of noetherian rings which is either flat with CM fibers or (essentially) lci, then for M a secantly CM R-module, $M \otimes_R S$ is secantly CM over S and the higher tor's vanishes. Similarly for R qe and φ $\hat{lci}$, where $\hat{lci}$ means that for ~~#~~ every $q \in \mathrm{Spec}(S)$ with contraction $p \subset R$, the homomorphism of cnlr $\hat{R_p} \longrightarrow \hat{S_q}$ has the property that for some (equivalently every) Cohen factorization $\hat{R_p} \xrightarrow{\psi} T \underset{\pi}{\twoheadrightarrow} \hat{S_q}$, ψ flat with regular fiber ring, $\ker(\pi)$ is generated by a regular sequence.

Given rings $A_i, i \in I$, I infinite (countable), let

$\prod' A_i$ be $\prod A_i / (\text{elements of finite support})$, similarly

$\prod' M_i$, where M_i is an A_i-module, is a $\prod' A_i$-module.

If $I = \mathbb{N}$ and all $A_i = A$ and $M_i = M$, we denote $\prod' A_i$ by $A^\diamond$ and $\prod' M_i$ by $M^\diamond$. If f is an element of A equipped with a compatible system of roots f^{1/p^n} for a prime p, let S_f be the multiplicative set in $A^{\mathbb{N}}$ consisting of $(f^{\varepsilon_0}, f^{\varepsilon_1}, \dots)$, $\varepsilon_i \in \mathbb{N}[\frac{1}{p}]$, $\varepsilon_i \to 0$. If ~~#~~ $B \to A$ is a homomorphism with B coherent (resp. noetherian) and M an A-module which is f^{1/p^∞}-almost flat (resp. faithfully flat) over B then $S_f^{-1} M^\diamond$ is flat (resp. faithfully flat) over B. For any ideal of finite type $I \subset A$, $M^\diamond$ and $S_f^{-1} M^\diamond$ are complete (not necessarily separated) for the I-adic filtration.

If R is a noetherian local ring with system of parameters $\underline{x}$ and B an R-algebra containing a compatible system of roots f^{1/p^n} of some $f \in B$, and B is f^{1/p^∞} almost big CM w.r.t. $\underline{x}$ over R, then $S_f^{-1} B^{\diamond}$ is an actual big CM R-algebra w.r.t. $\underline{x}$. This replaces in our approach the techniques of Hochster and Shimomoto used previously to construct big CM algebras in mixed characteristic, see [1].
[Alternatively, we can use ultrapowers instead of $B \mapsto B^{\diamond}$.]

By an integral Tate ring (itr) we mean a ring of definition of a Tate ring.
If B is an itr with pseudo-uniformizer b s.t. $p \in (b^p)$ for a prime p, and $\{f^{1/p^n}\}$ is a compatible system of roots, we say that B is f^{1/p^∞}-almost perfectoid iff the frobenius map $B/bB \to B/b^p B$ is an f^{1/p^∞}-almost isomorphism. This condition is independent of the choice of b and equivalent to the condition that the b-adic completion $(S_f^{-1} B^{\diamond})^{\wedge}$ is perfectoid.

The following is an updated form of the notion of a perfectoid ring:
For a fixed prime p, we say that a topological commutative ring A is perfectoid if the topology is defined by a complete separated exhaustive decreasing filtration $\mathrm{Fil}^{\gamma} A$, $\gamma \in \mathbb{Z}[\frac{1}{p}]$, s.t. $p \in \mathrm{Fil}^{r}$ for some $r > 0$ and the frobenius map $\mathrm{Fil}^{r} / \mathrm{Fil}^{r'} \longrightarrow \mathrm{Fil}^{pr} / \mathrm{Fil}^{pr'}$ is bijective whenever $r' \geq r$ and $r' - r$ small enough. For linearly topologized rings, if there is a perfectoid filtration one can choose it s.t. $A = \mathrm{Fil}^{0} A$.

This notion specializes (after proving certain things) to previous notions of perfectoid rings, in particular for non-archimedean fields we get perfectoid fields, for Banach algebras over a perfectoid field we get Scholze's notion, for Tate rings we get Fontaine's notion, for adic* rings we get the notion studied in [5].

The strong monomial conjecture (which implies the strong direct summand conjecture) states that if A is a noetherian local ring of dimension n and P a height 1 prime with $\dim(A/P) = n-1$, and $x_1, \ldots, x_n$ a system of parameters with $x_1 \in P$, then for $d_1, \ldots, d_n$ in $\mathbb{N}$

$$\xi = x_1^{d_1+1} x_2^{d_2} \cdots x_n^{d_n} \notin P(x_1^{d_1+1}, \ldots, x_n^{d_n+1}).$$

This can be reduced to the complete case and follows easily if there are big CM algebras $B_0 \to B_1$ over $A \to A/P$, which extends observations of Hochster. Thus the strong direct summand conjecture holds by results of [1] in the mixed characteristic case and previous techniques in the equal characteristic case.

Let $I = (x_1^{d_1+1}, \ldots, x_n^{d_n+1})$. Given $B_0 \to B_1$ consider

$$IB_0/I^2B_0 \simeq (B_0/I)^n \longrightarrow (B_1/IB_1)^n$$

We prove that $\xi \notin (PI + I^2)B_0$. Otherwise looking at the image of $[\xi]$ we get that

$x_2^{d_2} \cdots x_n^{d_n} \in IB_1$ which does not hold by the arguments for $\exists$ big CM module $\to$ monomial conjecture.

The derived variant of the monomial conjecture (dvmc), asserts that if A is a nlr of dim n with s.o.p. $x_1, \ldots, x_n$ and $X \xrightarrow[f]{} \mathrm{Spec}(A)$ is proper surjective, $d_i \in \mathbb{N}$, then the class of

$$x_1^{d_1} \quad x_n^{d_n} \text{ in } \mathbb{H}^0(X, K_\cdot(x_1^{d_1+1}, \quad , x_n^{d_n+1} ; \mathcal{O}_X))$$

is non zero.

For A regular this is equivalent to the derived variant of the direct summand conjecture of [2]. It is claimed that (dvmc) can be proved similarly.

If $R \subset S$ are rings let the p-integral closure of R in S $p\text{-}ic(R,S)$ be the smallest intermediate ring T s.t. $x^p \in T \Rightarrow x \in T$. When $R[\frac{1}{p}] \xrightarrow{\sim} S[\frac{1}{p}]$ this amounts to replacing $(S/p)_{red}$ by the subring of elements ~~sent~~ sent by a frobenius iterate to the image of R.

If A is a perfectoid integral Tate ring with pseudo-uniformizer b and $f \in A$ consider $B = A[f^{1/p^\infty}]$ and $B' =$ the p-ic of B in $B[\frac{1}{b}]$. Then the b-adic completion of B' is perfectoid and B'/b is faithfully flat over B/b. This is shown in [5] (and also for Bhatt's version with a monic polynomial) and refines an almost statement of Y. André.

If A is a perfectoid ring in the sense of [5] and $U \subset \mathrm{Spec}(A)$ an open subscheme containing $V(I)^c$ for I an ideal of definition, and m the radical ideal defining $\mathrm{Spec}(A) - U$, then we have a version of the purity theorem (Faltings, Scholze, Kedlaya-Liu) studied in [5]:

(finite étale $(A,m)^a$-algebras) $\longrightarrow$ (finite étale U-schemes)

is an equivalence of categories.

If B is a finite étale $(A,m)^a$-algebra (Scholze's terminology) then under the localization map A-algebras $\longrightarrow$ $(A,m)^a$-algebras B has perfectoid representatives, in particular it lifts canonically to $B_{!!}$, B_*, the image of $B_{!!} \to B_*$ and its integral closure in B_* (all topologized to be adic over A). The tilting equivalence holds for the above.

As a preliminary to our integral refinement of the perfectoid Abhyankar lemma, let A be a perfectoid itr with a pseudo-uniformizer b having a compatible system of roots b^{1/p^n} and $f \in A$ an element with a compatible system of roots. Then the perfectoid space $\mathrm{Spa}(A[\frac{1}{b}], ic(A))$ is covered by $U_0 : |f| \le |b^n|$

$U_1 : |f| \ge |b^n|$.

$\mathcal{O}(U_0) = \widehat{A[\frac{1}{b}]}\langle \frac{f}{b^n} \rangle$ and we replace $\mathcal{O}(U_0)^+$ by the slightly smaller perfectoid ring topologically generated by A and the $(f/b^n)^{1/p^k}$; denote it by $\mathcal{O}(U_0)^{p+}$, do similarly for U_1 and $U_{01} = U_0 \cap U_1$.

Then one has that the augmented Čech complex

$$0 \to A \to \mathcal{O}(U_0)^{p+} \oplus \mathcal{O}(U_1)^{p+} \to \mathcal{O}(U_{01})^{p+} \to 0$$

is $(f^{1/p^\infty}, b^{1/p^\infty})$ - almost exact.

This implies that $\forall i$

$$A/b^i \longrightarrow \text{"}\varprojlim_n\text{"}\ \mathcal{O}(|f| \geq |b^n|)^{p+}/b^i$$

is an f^{1/p^∞} - almost isomorphism.

Contrary to André, we do not have to work over a perfectoid field, we do not assume that f is a non-zero-divisor, and we replace $(fb)^{1/p^\infty}$ by f^{1/p^∞}. This allows one to extend the weak functoriality approach of Heitmann and Ma [3] to height 1 primes of residue characteristic p and is also used in the proof of Th. R(b).

If B is a finite étale $A[\frac{1}{f}]$ - algebra, one considers $f^{-1/p^\infty}A \subset A[\frac{1}{f}]$, the integral closure B_0 of $f^{-1/p^\infty}A$ in B. Then B_0 is b-adically complete separated, f^{1/p^∞} - almost perfectoid, and B_0/b^i is f^{1/p^∞} - almost finite étale over A/b^i $\forall i$, and almost faithfully flat if B is faithfully flat over $A[\frac{1}{f}]$.

Let R be a local noetherian ring s.t. $\mathrm{Spec}(R)$ is equipped with an fs log structure which is log regular in the sense of Kato. Using the above techniques one can extend the derived variant of the direct summand conjecture to such rings:

If $X \xrightarrow{f} \mathrm{Spec}(R)$ is proper surjective, then $R \xrightarrow{\alpha} R\Gamma(X, \mathcal{O}_X)$ has a left inverse in $D(R)$.

We may assume that there is a global chart and complete and possibly change the log structure so that R is of the form

$k[[P]]$ P a sharp fs monoid, k a field in the equal characteristic case

$I[[P]]/(h)$, I a Cohen ring, h mapping to the uniformizer p of I in the mixed characteristic $(0,p)$ case.

In characteristic 0 one can prove the result without reduction to char. p, namely one knows that there are desingularizations by log blow-ups $Y \xrightarrow{\pi} \mathrm{Spec}(R)$, for each such $R^i\pi_*\mathcal{O}_Y = 0$ $\forall i > 0$, $\pi_*\mathcal{O}_Y = \mathcal{O}$, and one can use a result of Kovács (extended to more general schemes).

When the residue field is of char. $p > 0$ WMA it is perfect. Let $R_\infty = \mathbb{Z}(P[\frac{1}{p}]) \underset{\mathbb{Z}_P}{\otimes} R$, which is $P^{gp}[\frac{1}{p}]/P^{gp}$-graded.

One can reduce to the case where f is generically finite étale, say over $\mathrm{Spec}(R[\frac{1}{h}])$, $h \neq 0$.

In the mixed characteristic case let A_∞ be the p-integral closure of $R_\infty[h^{1/p^\infty}]$ in $R_\infty[h^{1/p^\infty}][1/p]$.

To show that α has a left inverse it suffices to show that $\mathrm{Tor}_i^R(K, \alpha)$ is injective $\forall i$ for every bounded complex K of finite length R-modules.

Using perfectoid techniques one gets that the image of $\mathrm{Ker}\, \mathrm{Tor}_i^R(K, \alpha)$ in $\mathrm{Tor}_i^R(K, A_\infty)$ is h^{1/p^∞}-almost zero. To get what we want we use

<u>Proposition</u>. Let $I \subset R_\infty$ be an ideal s.t. $I A_\infty$ contains all the h^{1/p^i}, then there is a sequence $\gamma_n \in P[\frac{1}{p}]$, $\gamma_n \to 0$, s.t. $\underbrace{I \to gr_{[\gamma_n]}(R_\infty)}_{\text{the projection to the } [\gamma_n] \text{ component}}$ is surjective.

In equal characteristic p the ~~correspondin~~ counterpart is that $\exists\ c \in P$ s.t. for $n >> 0$ the $[c/p^n]$ graded component of h^{1/p^n} generates $gr_{[c/p^n]}(R_\infty)$ over R.

If $P \xrightarrow{\alpha} R$ is a chart for a log regular structure, P a toric monoid, (R a noetherian ring), for every $[c] \in P^{gp}_{\mathbb{Q}}/P^{gp}$ we have $N_{[c]} := (c + P^{gp}) \cap P_{\mathbb{Q}}$

and a reflexive rank one R-module $\mathcal{L}_{[c]} = \mathbb{Z} N_{[c]} \underset{\mathbb{Z}P}{\otimes} R$.

<u>Theorem R</u>: (a) If $P \xrightarrow{\alpha} R$ is log regular as above and S/R is finite tf generically étale, then the diagonal idempotent e in Γ(locus of étaleness, $S \underset{R}{\tilde{\otimes}} S$) comes from an element of

$$\sum_{c \in P^{gp}_{\mathbb{Q}}} (S^\vee \underset{R}{\otimes} \mathcal{L}_{[c]})^{\vee\vee} \underset{R}{\otimes} (S^\vee \underset{R}{\otimes} \mathcal{L}_{[-c]})^{\vee\vee} \quad (S^\vee = \mathrm{Hom}_R(S, R)).$$

(b) Same with e replaced by he where h is any element vanishing on the ramification locus and $P^{gp}_{\mathbb{Q}}$ replaced by $P^{gp}[\frac{1}{p}]$, where p is a prime s.t. R is a $\mathbb{Z}_{(p)}$-algebra.

Functorial big CM algebras.

For a fixed prime p consider the category $\mathcal{C}$ of complete noetherian local domains of mixed char $(0,p)$ and residue field with a finite p-rank, with morphisms being the ring homomorphisms (not ~~neces~~ necessarily local).

<u>Theorem F.</u> There is a functorial construction $A \mapsto B(A)$ with $B(A)$ a vnt secantly CM A-algebra.

Given A and finite subsets $T' \subset T \subset A - \{0\}$, consider

$$A_\infty = A[f^{1/p^\infty} \text{ for } f \in T].$$

Let $\hat{A}_\infty$ be the p-adic completion. Then we have a sheafy adic space $X = \mathrm{Spa}(\hat{A}_\infty[\frac{1}{p}], ic(\hat{A}_\infty))$, and on it $U \mapsto \mathcal{O}^+(U)^\wedge$ is at least a presheaf on the category of rational domains, and can be extended by $\varprojlim$ to more general opens. Let $U_{T'}$ be the open in X defined by the non vanishing of the functions in T'.

Then for some choices of T, T', $\Gamma(U_{T'}, \hat{\mathcal{O}}^+)$ is h^{1/p^∞}-almost balanced big CM, h being $\prod_{f \in T'} f$, and

$S_h^{-1}(\Gamma(U_{T'}, \hat{\mathcal{O}}^+)^\diamond)$ is vnt secantly CM, and also its p-adic completion.

Taking $\varinjlim$ over all choices of T, T' we get a functorial construction. In the talk the idea was to show the vnt property using weak local uniformization, but this is not needed now.

There are similar constructions for the category $\mathcal{C}'$ of mixed char $(0,p)$ henselian local domains which are ind-étale over rings of finite type, and for the category of mixed characteristic domains that are p-adic henselizations of rings of finite type.

In char. $p>0$, if R is an excellent reduced locally equidimensional $\mathbb{F}_p$-algebra and $h \in R$ a regular element vanishing on the non CM locus then R_{perf} is h^{1/p^∞}-almost CM (for localizations) and

$S_h^{-1} \hat{R}_{perf}$ is vnt secantly CM. Taking $\varinjlim$ over all h we get a functorial construction w.r.t. maximally dominating morphisms.

In the talk I discussed a procedure for reducing weak functoriality to the case of injective homomorphisms.

① Let X be an integral qc scheme. Then one has a very weak desingularization result that $\forall x \in X$ $\exists$ $h: X' \to X$ birational of ft s.t. X' has a regular point above x.

For $\mathcal{O}_{X,x}$ singular it suffices to take a 1 dimensional domain $\hat{\mathcal{O}}_{X,x}/P$ s.t. P is in the regular locus of the completion and blow up repeatedly the closed point of the proper transform.

② Let $R \twoheadrightarrow R/I$ be noetherian rings s.t. $\forall \mathfrak{p} \in \mathrm{Ass}(R/I)$, $IR_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence of length c.

One can find $f_1, \dots, f_c \in I$ which generate I at every $\mathfrak{p} \in \mathrm{Ass}(R/I)$, and g which is invertible at points of

$\mathrm{Ass}(R/I)$ s.t. $gI \subset (f_1, \dots, f_c)$. Then

$$R[f_1/g, \dots, f_c/g]/\left(\frac{f_1}{g}, \dots, \frac{f_c}{g}\right) \xrightarrow{\sim} R/I,$$ and the

f_i/g form a regular sequence in a neighborood of $\mathrm{Spec}(R/I)$.

③ If $R \longrightarrow S$ is a morphism of ft between qe domains, then there are domains of ft/R

$$S \subset S' \xleftarrow{\alpha} S'' \quad \text{s.t. } S'/S \text{ is finite and}$$

$\ker(\alpha)$ is generated by a regular sequence and $R \to S''$ is injective. Sometimes one has to repeat the proof in other settings.

The weak functoriality for chains of homomorphism of Y. André [1] can be extended to not necessarily local homomorphisms;

<u>Theorem WF</u>: If $A_0 \to A_1 \to \dots \to A_n$ are local (complete) noetherian domains of residue characteristic p with not nec. local morphisms, then there is a chain of Vnt secantly CM algebras $B_0 \to B_1 \to \dots \to B_n$ over the given chain.

Using the Artin-Rotthaus theorem one can show that each A_i is a filtered $\varinjlim$ of local rings (O_α) essentially of ft/$\mathbb{Z}$ s.t. $O_\alpha \to \hat{A}_i$ is lci and $O_\alpha \to O_\beta$ is ess. lci. Using this reduce the problem to a chain in $\mathcal{C}_0$, WMA A_0 is of mixed characteristic. Then one can construct

$$\begin{tikzcd}
 & A''_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A''_n \arrow[d] \\
 & A'_1 \arrow[r] & \cdots & A'_n \\
A_0 \arrow[r] \arrow[uur] \arrow[ur] & A_1 \arrow[u, phantom, "\cup"] & & A_n \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

s.t. A'_i/A_i is finite

A'_i is A''_i mod. a regular sequence

$A_0 \to A''_1 \to \dots \to A''_n$ has injective transition maps.

In characteristic 0 our ideals are similar to techniques of [6].

I claim that there is a functorial construction of unit secantly CM algebras over the category of locally equidim. reduced $\mathbb{Q}$-algebras of finite type with morphisms the ones with maximally dominating Spec.

Namely for $A/\mathbb{Q}$ we spread it out to $\mathcal{A}/\mathbb{Z}$ of f.t. and take

$$\prod_p{}' (\text{all } s_h)^{-1} (\mathcal{A}_{\bar{p}})^{\diamond}_{perf} .$$

One obtains also an analogue of Th. WF,

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