This report contains observations made after the conference. Y. Andre mentioned [6] and L. Ma sent [4].

There are several versions of the definition of big Cohen-Macaulay modules. Let R be a noetherian local ring of dimension d, we say that an R-module M is CM iff depth \( M = d \) (depth defined by local cohomology). One says that M is CM w.r.t. a system of parameters \( (x_1, \ldots, x_d) = X \) iff \( X \) is a regular sequence on M (in the sense of EGA, i.e. without non-triviality assumption) and \( M \neq X \cdot M \). One says that M is balanced big CM if it is CM w.r.t. any s.o.p. of R.

For a noetherian ring R we say that an R-module M is secondly CM iff every strictly secant sequence \( x_1, \ldots, x_d \) of R is M-regular. This is equivalent to the condition that \( \text{depth} (M_p) = \dim \tilde{R}_p \) or \( \infty \) \( \forall p \in \text{Spec}(R) \). [There is an almost variant.]

One says that an R-module M is valuatorily non trivial (unitl) if for every homomorphism \( R \rightarrow W \) to a valuation ring W with residue field \( k \), \( (M \otimes W) / (\text{torsion}) \otimes W \neq 0 \).

For M an R-algebra B this is equivalent to saying that \( \text{Spec}(B) \rightarrow \text{Spec}(R) \) is covering for the \( v \)-topology, or that \( R \rightarrow B \) is strongly submersive in the sense of Nagata.

If an R-module M over a complete local noetherian domain is solid in the sense of [4] then it is vnt. Any CM module over a complete local noetherian domain is solid.

Over a noetherian local \( R \), if M is CM, \( \hat{M} \) is balanced CM.

If R is catenarian and equidimensional, secondly CM is the same as balanced CM without the non-triviality condition.

If M is secondly CM and vnt over R then R is locally equidimensional and universally catenarian.

Claim. If R is noetherian, \( \text{radically complete (separated)} \), x non zero divisor on R and on a module M, and \( M / x \cdot M \) is secondly CM over \( R / x \), then \( \hat{M} \) is secondly CM over \( R \). [There is an almost variant.]
If \( R \rightarrow S \) is a homomorphism of noetherian rings which is either flat with CM fibers or (essentially) lci, then for \( M \) a secondly CM \( R \)-module, \( M \otimes_S S') \) is secondly CM over \( S \) and the higher tor's vanishes. Similarly for \( R \otimes \mathfrak{p} \) and \( \mathfrak{p} \) lci, where lci means that for every \( \mathfrak{p} \in \text{Spec}(S) \) with contraction \( f : CR \rightarrow \hat{S} \), the homomorphism of Cohen factorization \( R_f \rightarrow T \rightarrow \hat{S} \), \( T \) flat with regular fiber ring, \( \ker(T) \) is generated by a regular sequence.

Given rings \( A_i, i \in I \), \( I \) infinite (countable), let \( \prod_i^\wedge A_i \) be \( \prod_i A_i/(\text{elements of finite support}) \), similarly \( \prod_i^\wedge M_i \), where \( M_i \) is an \( A_i \)-module, is a \( \prod_i^\wedge A_i \)-module.

If \( I = N \) and all \( A_i = A \) we denote \( \prod_i^\wedge A_i \) by \( A^\wedge \) and \( \prod_i^\wedge M_i \) by \( M^\wedge \). If \( f \) is an element of \( A \) equipped with a compatible system of roots \( f^{[p]} \) for a prime \( p \), let \( Sp \) be the multiplicative set in \( A \), consisting of \( \left( f^{[p]}, f^{[p]}, \ldots \right) \), \( e_i \in \mathbb{N}^[\wedge p] \), \( e_i \rightarrow 0 \). If \( f : A \rightarrow B \) is a homomorphism with \( B \) coherent (resp. noetherian) and \( M \) an \( A \)-module which is flat (resp. faithfully flat) over \( B \), then \( Sp^{-1} M^\wedge \) is flat (resp. faithfully flat) over \( B \). For any ideal of finite type \( I \subset A \), \( M^\wedge \) and \( Sp^{-1} M^\wedge \) are complete (not necessarily separated) for the I-adic filtration,
If $R$ is a noetherian local ring, with system of parameters $x$ and $B$ an $R$-algebra containing a compatible system of roots $f^{1/p_n}$ of some $f \in B$ and $B$ is $f^{1/\infty}$-almost big CM over $R$, then $S_f B^\wedge$ is an actual big CM $R$-algebra w.r.t. $x$. This replaces in our approach the techniques of Hochster and Shimomura used previously to construct big CM algebras in mixed characteristic, see [1].

[Alternatively, we can use ultrapowers instead of $B \to B^\wedge$.]

By an integral Tate ring (itr) we mean a ring of definition of a Tate ring.

If $B$ is an itr with pseudo-uniformizer $b$ s.t. $p \notin (b^r)$ for a prime $p$, and $f^{1/p_n}$ is a compatible system of roots, we say that $B$ is $f^{1/\infty}$-almost perfectoid iff the Frobenius map $B/\mathfrak{m} B \to B/\mathfrak{m}^r B$ is an $f^{1/\infty}$-almost isomorphism. This condition is independent of the choice of $b$ and equivalent to the condition that the $b$-adic completion $(S_f B^\wedge)$ is perfectoid.

The following is an updated form of the notion of a perfectoid ring:

For a fixed prime $p$, we say that a topological commutative ring $A$ is perfectoid if the topology is defined by a complete separated exhaustive decreasing filtration $\text{Fil}_p A$, $y \in \mathbb{Z} [\frac{1}{p}]$, $s, t \in \mathbb{Z}$ for some $y \geq 0$ and the Frobenius map $\text{Fil}_p^r (\text{Fil}_p^s) \to \text{Fil}_p^{s'p^{r'}}$ is bijective whenever $y \geq y'$ and $p' - p$ small enough. For linearly topologized rings, if there is a perfectoid filtration one can choose it s.t. $A = \text{Fil}_p^0 A$. 
This notion specializes (after proving certain things) to previous notions of perfectoid rings, in particular for non-Archimedean fields we get perfectoid fields, for Banach algebras over a perfectoid field we get Scholze's notion, for Tate rings we get Fontaine's notion, for adic rings we get the notion studied in [5].

The strong monomial conjecture (which implies the strong direct summand conjecture) states that if $A$ is a noetherian local ring of dimension $n$ and $P$ a height 1 prime with $\dim(N_P) = n-1$, and $x_1, \ldots, x_n$ a system of parameters with $x_i \in P$, then for $d_1, \ldots, d_n \in \mathbb{N}$

$$s = x_1^{d_1+1} x_2^{d_2} \ldots x_n^{d_n} \in P(x_1^{d_1+1}, \ldots, x_n^{d_n+1}),$$

This can be reduced to the complete case and follows easily if there are big CM algebras $B_0 \rightarrow B_1$, one $A \rightarrow A/P$ which extends observations of Hochster. Thus the strong direct summand conjecture holds by results of [1]

in the mixed characteristic case and previous techniques in the equal characteristic case.

Let $I = (x_1^{d_1+1}, \ldots, x_n^{d_n+1})$. Given $B_0 \rightarrow B_1$ consider

$$IB_0 / I^2 B_0 \sim (B_0/I) \xrightarrow{n} (B_1/I B_1)^n$$

We prove that $s \in (P + I^2)B_0$, otherwise looking at the image of $[5]$, we get that $x_1^{d_1+2} \ldots x_n^{d_n} \in IB_1$, which does not hold by the arguments for an big CM module $\rightarrow$ monomial conjecture.
The derived variant of the monomial conjecture (dvmc), asserts that if $A$ is a rlr of dim $n$ with s.o.p $x_1, \ldots, x_n$ and $X \to \text{Spec}(A)$ is proper surjective, $d_i \in \mathbb{N}$ then the class of

$$x_1^{d_1} \cdots x_n^{d_n} \text{ in } \mathcal{H}^0(X, k, (x_1^{d_1 + 1}, \ldots, x_n^{d_n + 1}, \Theta_X))$$

is non zero.

For a regular this is equivalent to the derived variant of the direct summand conjecture of [21].

It is claimed that (dvmc) can be proved similarly.

If $R \subseteq S$ are rings let the $p$-integral closure of $R$ in $S$ $p$-ic$(R, S)$ be the smallest intermediate ring $T$ s.t. $x \in T \Rightarrow x^p \in T$, when $R[\frac{1}{p}] \to S[\frac{1}{p}]$ this amounts to replacing $(S/p)$ and by the ordering of elements sent by a Frobenius iterate to the image of $R$.

If $A$ is a perfectoid Tate ring with pseudo-uniformizer $b$ and $f \in A$ consider $B = A[\frac{1}{f^{\nu b^1}}]$ and $B' = \text{p$-$ic$ of } B \text{ in } B[\frac{1}{b}]$. Then the $b$-adic completion of $B'$ is perfectoid and $B'/b$ is faithfully flat over $B/b$, this is shown in [5] (and also for Bhut's version with a monic polynomial) and refines an almost statement of V. André.
If $A$ is a perfectoid ring in the sense of [5] and $U \subseteq \text{Spec}(A)$ an open subscheme containing $V(I)^c$ for $I$ an ideal of definition, and in the radical ideal defining $\text{Spec}(A) - U$, then we have a version of the purity theorem (Faltings, Scholze, Kedlaya-Liu) studied in [5];

$(\text{finite étale } (A, m)^a - \text{algebras}) \rightarrow (\text{finite étale } U\text{-schemes})$

is an equivalence of categories.

If $B$ is a finite étale $(A, m)^a$-algebra (Scholze's terminology) then under the localization map $A\text{-algebras} \rightarrow (A, m)^a\text{-algebras}$ $B$ has perfectoid representatives, in particular it lifts canonically to $B!! \rightarrow B_*$, the image of $B!! \rightarrow B_*$ and its integral closure in $B_*$ (all topologized to be adic over $A$).

The tilting equivalence holds for the above.

As a preliminary to our integral refinement of the perfectoid Abhyankar lemma, let $A$ be a perfectoid $A$-algebra with a pseudo-uniformizer $b$ having a compatible system of roots $b_1, \ldots, b_n$ and $f \in A$ an element with a compatible system of roots. Then the perfectoid space $\text{Spa}(A[1/b], \mathfrak{m}(A))$ is covered by $U_0 : |f| \leq |b^n|$ and $U_1 : |f| > |b^n|$. $\mathcal{O}(U_0) = \langle \frac{f}{b^n} \rangle$ and we replace $\mathcal{O}(U_0)^+$ by the slightly smaller perfectoid ring topologically generated by $A$ and the $(f/b^n)^{1/b^n}$; denote it by $\mathcal{O}(U_0)^+$, etc. Similarly for $U_1$ and $U_{01} = U_0 \cap U_1$. 
Then one has that the augmented Čech complex

\[
0 \to A \to O(U_0)^{\mathbb{p}^+} \otimes O(U_1)^{\mathbb{p}^+} \to O(U_{0,1})^{\mathbb{p}^+} \to 0
\]

is \((f^{\mathcal{P}_1}, b^{\mathcal{P}_0})\)-almost exact.

This implies that \(A_i\)

\[
\frac{A_i}{b_i} \longmapsto \lim_{n \to \infty} O(1f_{\mathcal{P}} = b^i)^{\mathbb{p}^+}/b_i
\]

is an \(f^{\mathcal{P}_1}\)-almost isomorphism.

Contrary to André, we do not have to work over a perfect field, we do not assume that \(f\) is a non-zero-divisor, and we replace \((fb)^{\mathcal{P}_0}\) by \(f^{\mathcal{P}_1}\). This allows one to extend the weak functoriality approach of Heitmann and Ma to height 1 primes of residue characteristic \(p\) and is also used in The R(b).

The proof of

If \(B\) is a finite étale \(A[\frac{1}{f}]\)-algebra, one considers \(f^{\mathcal{P}_1} A \subseteq A[\frac{1}{f}]\), the integral closure \(B_0 \subseteq f^{\mathcal{P}_1} A\) in \(B\). Then \(B_0\) is \(f^{\mathcal{P}_1}\)-almost complete separated, \(f^{\mathcal{P}_1}\)-almost perfectoid, and \(B_0/b_i\) is \(f^{\mathcal{P}_1}\)-almost finite étale over \(A/f_i, b_i\); and almost faithfully flat if \(B\) is faithfully flat over \(A[\frac{1}{f}]\).

Let \(R\) be a local noetherian ring s.t. \(\text{Spec}(R)\) is equipped with an fs log structure which is log regular in the sense of Kato. Using the above
techniques one can extend the derived variant of the direct summand conjecture to such rings;

If $X \to \text{Spec}(R)$ is proper surjective, then $R \to R^\wedge(X, \mathcal{O}_X)$ has a left inverse in $\mathcal{D}(R)$.

We may assume that there is a global chart and complete and possibly change the log structure so that $R$ is of the form $R[\{P, q\}]$ $P$ a sharp fs monoid, $k$ a field in the equal characteristic case

$T[\{P, q\}] / (h)$, $T$ a Cohen ring, $R$ mapping to the uniformizer of $T$ in the mixed characteristic $(0, p)$ case.

In characteristic 0 one can prove the result without reduction to char. $p$, namely one knows that there are desingularizations by log blow-ups $Y \xrightarrow{\pi} \text{Spec}(R)$, for each such $R^\wedge \times \mathcal{O}_Y = 0$ $h > 0$, $\nu^* \mathcal{O}_Y = 0$, and one can use a result of Kawamata (extended to more general schemes).

When the residue field is of char. $p > 0$ WMA it is perfect, let $R_{\infty} = \mathbb{Z}(P[\{q\}]) \otimes \mathbb{Z}[\frac{1}{p}]$, which is $p$-graded.

One can reduce to the case where $f$ is generically finite étale, say over $\text{Spec}(R[\{q\}])$, $h \neq 0$.

In the mixed characteristic case let $A_{\infty}$ be the $p$-integral closure of $R_{\infty}[\frac{1}{p}]$ in $R_{\infty}[\frac{1}{p}]$. 
To show that $a$ has a left inverse it suffices to show that $\text{Tor}_i^R(K, a)$ is injective for every bounded complex $K$ of finite length $R$-modules. Using perfection techniques one gets that the image of $\ker \text{Tor}_i^R(K, 2)$ in $\text{Tor}_i^R(K, R_{oo})$ is $\mathfrak{p}^{1/\mathfrak{p}^{\infty}}$-almost zero. To get what we want we use

**Proposition.** Let $I \subset R_{oo}$ be an ideal s.t. $I R_{oo}$ contains all the $\mathfrak{p}^{1/\mathfrak{p}^{n}}$, then there is a sequence $y_m \in P_{[p]}$, $y_m \rightarrow 0$, s.t. $\left[I \rightarrow (R_{oo}) (R_{oo}) \right]$ is surjective.

The projection to the $i$th component.

In equal characteristic $p$ the counterpart is that $\exists c \in P$ s.t. for $n \gg 0$ the $[c/p^n]$ graded component of $\mathfrak{p}^{1/\mathfrak{p}^{n}}$ generates $y_{[c/p^n]}(R_{oo})$ over $R$.

If $P \rightarrow R$ is a chart for a log regular structure, $P$ a toric monoid, then for every $c, h \in P_{qp}/p^{qp}$ we have $N_{c, h} = (c + p^{qp})P_{h}$

and a reflexive rank one module $L_{[c]} = \mathbb{Z} N_{[c]} \otimes R_{oo}$.

**Theorem 1.** If $P \rightarrow R$ is log regular as above and $S/R$ is finite generically étale, then the diagonal idemptotent $e$ in $P(S \otimes S)$ comes from an element of $S \otimes \mathbb{L}_{S/R} \otimes (S \otimes \mathbb{L}_{S/R}[-c])^\vee (S = \text{Hom}_R(S, R))$.

(a) With $e$ replaced by $h$, where $h$ is any element vanishing on the ramification locus $P_{\text{qp}}$ replaced by $P_{\text{qp}}[\mathfrak{p}]$, where $p$ is a prime s.t. $R$ is a $\mathbb{Z}_{(p)}$-algebra.
Functorial big CM algebras.
For a fixed prime $p$ consider the category $\mathcal{C}$ of complete noetherian local domains of mixed char $(0, p)$ and residue field with a finite $p$-rank, with morphisms being the ring homomorphisms (not necessarily local).

Theorem 1. There is a functorial construction $A \rightarrow B(A)$ with $B(A)$ a unique weakly CM $A$-algebra.

Given $A$ and finite subsets $T \subseteq T \subseteq A$, consider
$$\hat{A}_0 = A \left[ \frac{f}{y_0} \text{ for } f \in T \right].$$

Let $\hat{A}_0$ be the $p$-adic completion. Then we have a sheafy adele space $X = \text{Spa} \left( \hat{A}_0, \frac{1}{p}, \text{ic}(\hat{A}_0) \right)$, and on it $U_1 \rightarrow O^+ (U)$ is at least a presheaf on the category of rational domains, and can be extended by lim to more general opens. Let $U_{T'}$ be the open in $X$ defined by the non-vanishing of the functions in $T'$.

Then for some choices of $T, T'$, $P(U_{T'}, O^+)$ is balanced big.

$h$ for almost CM, $h$ being $T^f$, and $S^{-1} (P(U_{T'}, O^+))$ is unique weakly CM, and also its $p$-adic completion.

Taking lim over all choices of $T, T'$ we get a functorial construction. In the talk the idea was to show the unique property using weak local uniformization, but this is not needed now.

There are similar constructions for the category $G' / G$ of mixed char $(0, p)$ Henselian local domains which are ind-stable over rings of finite type, and for the
Category of mixed characteristic domains that are $\mathfrak{p}$-adic henselizations of rings of finite type.

In char. $p > 0$, if $R$ is an excellent reduced locally equidimensional $\mathbb{F}_p$-algebra and $h \in R$ a regular element vanishing on the non CM locus then $R_{\text{perf}}$ is $h^{\mathfrak{p}_{\text{perf}}}$-almost CM (after localizations) and

$\mathcal{S}_h R_{\text{perf}}$ is not densely CM. Taking $\mathcal{S}_h$ over all $h$, we get a functorial construction w.r.t. maximally dominating morphisms.

In the talk I discussed a procedure for reducing weak functoriality to the case of injective homomorphisms.

1. Let $X$ be an integral qc scheme. Then one has a very weak desingularization result that $\forall x \in X, x \rightarrow X' \rightarrow X$ birational of $f \circ x$.

$x'$ has a regular point above $x$.

For $\mathbb{O}_X$, singular it suffices to take a 1-dimensional domain $\mathbb{O}_X^\wedge / \mathfrak{p}$ s.t. $\mathfrak{p}$ is in the regular locus of the completion and blow up repeatedly the closed point of the proper transform.

2. Let $R \rightarrow R[I]$ be noetherian rings s.t. $\forall p \in \text{Ass}(R[I])$, $I_p \cap \mathfrak{p}$ is generated by a regular sequence of length $c$. One can find $f_1, \ldots, f_c \in I$ which generate $I$ at every $p \in \text{Ass}(R[I])$, and $g$ which is invertible at points of
\textbf{Ass (R/I) s.t. } g I \subset (f_1, \ldots, f_r). \text{ Then } \\
R[R_{f_1}, \ldots, f_i/g] / (\frac{f_i}{g}, \ldots, \frac{f_r}{g}) \cong R/I, \text{ and the } \\
f_i/g \text{ form a regular sequence in a neighborhood of } \text{Spec}(R/I). \\

3. \text{ If } R \rightarrow S \text{ is a morphism of } \text{finitely } \text{generated } \text{algebras, then there are domains } \text{of } \text{finite type } S \subset S' \subset S'' \text{ such that } S'/S \text{ is finite and } \\
\ker(\phi) \text{ is generated by a regular sequence and } R \rightarrow S'' \text{ is injective, sometimes one has to repeat the proof in other settings.} \\
The weak factoriality for chains of homomorphisms of \text{finitely generated } \text{algebras} \text{ can be extended to not necessarily } \text{local homomorphisms; } \\

\underline{Theorem WF:} \text{ If } A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \text{ are } \text{local } \text{noetherian } \text{domains of } \text{residue characteristic } p \text{ with } \text{not necessarily } \text{local } \text{morphisms,} \\
\text{then there is a chain of } \text{finitely generated } \text{algebras } B_0 \rightarrow B_1 \rightarrow \ldots \rightarrow B_n \text{ over the given chain.} \\

\text{Using the Artin-Rostovsky theorem one can show that each } \\
\text{A}_i \text{ is a filtered } \text{lim of } \text{local rings essentially of finitely generated } \text{algebras } \\
\text{s.t. } A_0 \rightarrow A_1 \text{ is } \text{local} \text{ and } \\
A_0 \rightarrow \text{O}_0 \text{ is } \text{local}, \text{ \text{Using this reduce the problem} \\
to a chain in } \text{O}_0, \text{ with } A_0 \text{ is of mixed characteristic. Then } \\
\text{one can construct } \\
\text{A}_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \\
\text{s.t. } A_i / A_{i-1} \text{ is } \text{finite } \\
A_i / A_{i-1} \text{ is a regular sequence } \\
A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \text{ has injective transition maps,}
In characteristic 0 our ideals are similar to techniques of [6].

I claim that there is a functorial construction of non-separably CM algebras over the category of locally Euclidean, reduced $\mathbb{Q}$-algebras of finite type with morphisms the ones with maximally dominating Spec.

Namely for $A$ we spread it out to $A_\mathbb{Q} / \mathbb{Z}$ of fl., and take $\lim_{\rightarrow} \text{TT}^{(\text{all}\,\mathfrak{p})}(A_\mathbb{Q})_{\mathfrak{p}}$.

One obtains also an analogue of Th. WF.


