# SINGULARITIES MOD $p$, AND SINGULARITIES IN MIXED CHARACTERISTIC

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## 1. Disclaimer

This is the transcription of a lecture given by Karl Schwede for the MSRI Hot Topics Workshop on the homological conjectures. Any errors or typos are my responsibility[1].

This lecture concerns singularities in characteristic zero, followed by singularities in mixed characteristic, and how big Cohen-Macaulay algebras will play into these areas going forward. This is joint work with Linquan Ma.

## 2. Characteristic 0

Suppose $X$ is a normal variety over $\mathbf{C}$, and say $x \in X$ is a singularity. One of the first questions one tends to ask is whether $x$ is a rational singularity.

**Definition 2.1.** If we look locally, we can say $X = \operatorname{Spec} \mathscr{O}_{X,x}$, and take

$$\widetilde{X} \xrightarrow{\pi} X$$

a resolution of singularities. Then for our purposes, $X$ has rational singularities if

$$R\pi_* \mathscr{O}_{\widetilde{X}} = \mathscr{O}_X.$$

More concretely, this is same as saying that $X$ is normal (i.e. $\pi_* \mathscr{O}_{\widetilde{X}} = \mathscr{O}_X$) and the higher direct images vanish.

**Example 2.2.** Say $\mathcal{L}$ is a locally free sheaf of finite rank on a projective variety $X$ with rational singularities (locally, as in the definition). Then $H^i(X, \mathscr{L}) = H^i(\widetilde{X}, \pi_* \mathscr{L})$. In other words, with rational singularities, you can compute cohomology of a locally free sheaf by doing it on the resolution.

Here's another definition, which is maybe a bit easier to play with and for us, will be more useful.

**Definition 2.3** (Kempf [8])**.** $X$ has rational singularities if $X$ is Cohen-Macaulay and

$$\pi_* \omega_{\widetilde{X}} = \omega_X,$$

where $\omega$ is the dualizing sheaf.

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**Remark 2.4.** This is a useful characterization, because even if $X$ doesn't have rational singularities, we always have

$$R\pi_*\omega_{\widetilde{X}} = \pi_*\omega_{\widetilde{X}}$$

in characteristic 0.

There is a dual version, in the sense of local duality, which says that we can replace the condition $\pi_*\omega_{\widetilde{X}} = \omega_X$ by the condition that

$$H^d_{\mathfrak{m}}(\mathscr{O}_{X,x}) \to \mathbf{H}^d_{\mathfrak{m}}((R\pi_*\mathscr{O}_X)_x)$$

is an isomorphism, or really it's enough to show that this is injective.

**Example 2.5.** One can check that $A = \mathbf{C}[x, y, z]/(x^n + y^n + z^n)$ has rational singularities if and only if $n \leq 2$. In fact this is easy to compute, because one can blow up the singularities and get concrete descriptions of the dualizing sheaves.

The following theorem of Elkik serves as motivation for what we will do in the rest of the talk.

**Theorem 2.6** (Elkik [4])**.** *Suppose* $(A, \mathfrak{m})$ *is a local Noetherian ring, essentially of finite type over* $\mathbf{C}$*, and say* $f \in \mathfrak{m}$ *is a nonzerodivisor. If* $A/(f)$ *has rational singularities, then* $A$ *does as well.*

**Remark 2.7.** This basically says that if a variety $X$ has a singularity $x \in X$, then to show that it's rational you can just choose a hypersurface going through $x$ and show that it has a rational singularity at $x$.

*Proof.* Let $X = \operatorname{Spec} A$ and fix a resolution $\widetilde{X} \xrightarrow{\pi} X$. Let $\overline{H} = \pi^*V(f)$ be the scheme theoretic pullback, and let $\widetilde{H}$ resolve $\overline{H}$. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & A \arrow[r] \arrow[d] & A/(f) \arrow[r] \arrow[d] & 0 \\
 & R\pi_*\mathscr{O}_{\widetilde{X}} \arrow[r, "f"] & R\pi_*\mathscr{O}_{\widetilde{X}} \arrow[r] & R\pi_*\mathscr{O}_{\overline{H}} \arrow[d] & \\
 & & & R\pi_*\mathscr{O}_{\widetilde{H}}, &
\end{tikzcd}$$

Now using local cohomology and vanishing we get the following diagram with exact rows:

$$\begin{tikzcd}
H^{d-1}_{\mathfrak{m}}(A/(f)) \arrow[r] \arrow[d, hook] \arrow[dd, bend right=60, "\sim"'] & H^d_{\mathfrak{m}}(A) \arrow[r, "f"] \arrow[d] & H^d_{\mathfrak{m}}(A) \arrow[d] \\
\mathbf{H}^{d-1}_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\overline{H}}) \arrow[r, hook] \arrow[d] & \mathbf{H}^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}}) \arrow[r, "f"] & \mathbf{H}^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}}) \\
H^{d-1}_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{H}}) & &
\end{tikzcd}$$

We know $A/(f)$ has rational singularities, so $H^{d-1}_{\mathfrak{m}}(A/(f)) \to \mathbf{H}^{d-1}_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\overline{H}})$ is injective. Furthermore the dual of Grauer-Riemann vanishing tells us that the lower local cohomology of $R\pi_*\mathscr{O}_{\widetilde{X}}$ vanishes, so $\mathbf{H}^{d-1}_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\overline{H}}) \to \mathbf{H}^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}})$ is injective.

Now take a nonzero class $Z \in H^d_{\mathfrak{m}}(A)$, and assume it goes to 0 under the map $H^d_{\mathfrak{m}}(A) \xrightarrow{f} H^d_{\mathfrak{m}}(A)$ (we can do this since all elements in $H^d_{\mathfrak{m}}(A)$ are torsion). But by exactness $Z$ comes from a nonzero element in $H^{d-1}_{\mathfrak{m}}(A/(f))$. By injectivity, its image in $\mathbf{H}^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}})$ can't be zero. This concludes the proof, along with the fact that $A/(f)$ Cohen-Macaulay $\implies$ $A$ Cohen-Macaulay. □

## 3. Characteristic $p$

The following analogous definition is due to Karen Smith.

**Definition 3.1** (Smith [11])**.** Now let $(A, \mathfrak{m})$ be a excellent local Noetherian ring in characteristic $p > 0$. This is said to have $F$-rational singularities if

(1) $A$ is Cohen-Macaulay and
(2) If $N \subseteq H^d_{\mathfrak{m}}(A)$ is such that $F(N) \subseteq N$ ($F$ is Frobenius), then either $N = 0$ or $N = H^d_{\mathfrak{m}}(A)$.

**Remark 3.2.** Why is this useful? Say we have $X = \operatorname{Spec} A$ as in the definition and a birational map $\widetilde{X} \to X$, which could be a resolution of singularities, for example. Then we get a map

$$\varphi : H^d_{\mathfrak{m}}(A) \to \mathbf{H}^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}}).$$

Let $K = \ker\varphi$. Note Frobenius is compatible with this map, so whatever the kernel is automatically satisfies condition (2), i.e. $F(K) \subseteq K$, so then either $K = 0$ or everything. But since $\widetilde{X} \to X$ is birational $\varphi$ can't be zero, so $\varphi$ is injective.

We now have a powerful theorem which lets us reduce mod $p$ and check rationality of singularities.

**Theorem 3.3** (Smith [11])**.** *Say $A$ is a variety over $\mathbf{Q}$ in characteristic 0. Suppose that after reduction to characteristic $p >> 0$, we get $A_p$ having $F$-rational singularities. Then $A$ has rational singularities.*

**Theorem 3.4** (Hara [5], Mehta-Srinivas [10])**.** *The converse holds on a Zariski open set of $p$ in* $\operatorname{Spec}\mathbf{Z}$.

Furthermore, we have an analog of Elkik's theorem in characteristic $p$.

**Theorem 3.5** (Fedder-Watanabe)**.** *If $(A,\mathfrak{m})$ is local Noetherian in characteristic $p$ and $A/(f)$ has $F$-rational singularities, then so does $A$.*

## 4. Mixed Characteristic

We want analogs of these theorems in mixed characteristic. We'll use (integral perfectoid) big Cohen-Macaulay algebras.

**Definition 4.1.** Say $(A,\mathfrak{m})$ is a complete local Noetherian domain of mixed characteristic $(0,p)$. Then $A$ has BCM-rational singularities if

(1) $A$ is Cohen-Macaulay
(2) $H^d_{\mathfrak{m}}(A) \to H^d_{\mathfrak{m}}(B)$ is injective for all big Cohen-Macaulay $A$-algebras $B$.

**Remark 4.2.** In characteristic $p$, BCM-rational is the same as $F$-rational. In characteristic $p$, note that $F$-rationality can be computed on a computer, which are implemented in Macaulay 2! So if you are in mixed characteristic $p$, and you are lucky enough to have something to mod out by, you can just run a computer program and prove BCM-rationality.

This looks similar to the previous definitions, but now we replace the $R\pi_*\mathscr{O}_{\widetilde{X}}$ with a big Cohen-Macaulay algebra. But there is a fundamental reason why this is a good idea: in chaacteristic zero, $R\pi_*\mathscr{O}_{\widetilde{X}}$ is "like a big Cohen-Macaulay algebra", in the sense that

$$\mathbf{H}^i_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}}) = 0 \text{ for all } i < d.$$

It's worth mentioning that we have no idea whether this is independent of the choice of big (integral perfectoid?) Cohen-Macaulay algebra.

However, we do know the following.

**Theorem 4.3** (Ma-Schwede)**.** *For all birational $\pi : \widetilde{X} \to X = \operatorname{Spec} A$, there exists $B$, a big Cohen-Macaulay $A$-algebra, such that*

$$\ker(H^d_{\mathfrak{m}}(A) \to H^d_{\mathfrak{m}}(B)) \supseteq \ker(H^d_{\mathfrak{m}}(A) \to H^d_{\mathfrak{m}}(R\pi_*\mathscr{O}_{\widetilde{X}})).$$

So we really do want to work with big Cohen-Macaulay algebras: BCM rational implies rational in any other sense you want.

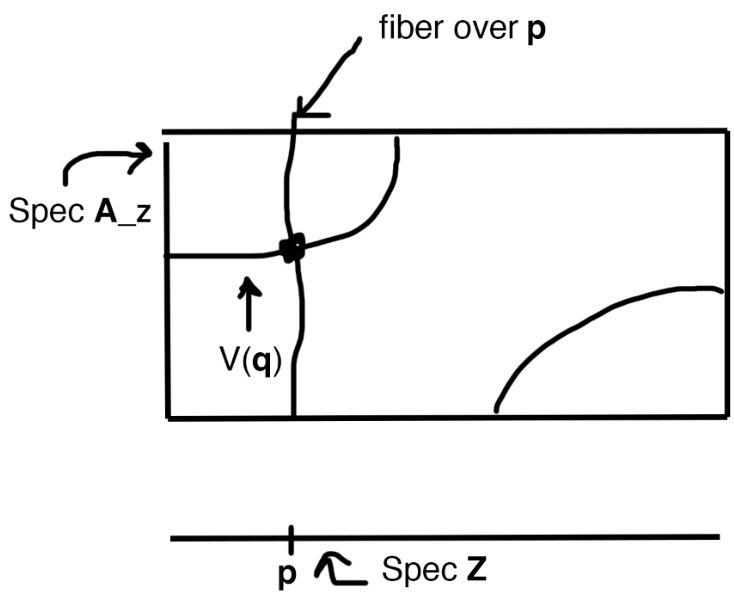
**Theorem 4.4** (Ma-Schwede)**.** *Suppose $(A,\mathfrak{m})$ is a complete local Noetherian ring of mixed characteristic $(0,p)$, and say $f \in \mathfrak{m}$ is a nonzerodivisor. If $A/(f)$ has BCM-rational singularities, then $A$ does as well.*

*Proof.* The proof is virtually the same as that of Theorem 2.6, except that instead of just having existence of big Cohen-Macaulay algebras, you need weakly functorial big Cohen-Macaulay algebras. Then we use a similar diagram like

$$\begin{tikzcd} H^{d-1}_{\mathfrak{m}}(A/(f)) \arrow[r] \arrow[d, hook] & H^d_{\mathfrak{m}}(A) \arrow[r, "f"] \arrow[d] & H^d_{\mathfrak{m}}(A) \arrow[d] \\ H^{d-1}_{\mathfrak{m}}(B/fB) \arrow[r, hook] & H^d_{\mathfrak{m}}(B) \arrow[r, "f"] & H^d_{\mathfrak{m}}(B) \end{tikzcd}$$

and repeat the proof basically word for word. $\square$

**Theorem 4.5** (Ma-Schwede)**.** *If $\mathfrak{q} \in \operatorname{Spec} A$ for $A$ a ring of finite type over $\mathbf{Q}$, and we take a model $A_{\mathbf{Z}}$ over $\mathbf{Z}$ and a corresponding prime $\mathfrak{q}_{\mathbf{Z}}$, then if $p \in \mathbf{Z}$ is such that $(p) + \mathfrak{q}_{\mathbf{Z}} \neq A_{\mathbf{Z}}$ and $(A_{\mathbf{Z},\mathfrak{q}_{\mathbf{Z}}})/p$ is $F$-rational, then $A_{\mathfrak{q}}$ is rational.*




Pictorially, this says that if the point in the intersection in the above picture has $F$-rational singularities, then in fact this point in mixed characteristic is BCM-rational. Then after further localization, it's also rational in characteristic 0.

One can define a test ideal thing for BCM algebras

$$\tau_B(R, \Delta),$$

and you get $\tau_B(R, \Delta) \subseteq J(R, \Delta)$. You also get a restriction theorem, which is basically a jazzed up version of Elkik's theorem.

You also get transformation rules under finite maps as well, which give you "arithmetic" applications, which we briefly discuss now. In particular, if $A/f$ is BCM-regular and $A$ is $q$-Gorenstein, then $(A, f^{1-\epsilon})$ is BCM-regular for all $\epsilon \in (0, 1)$. This implies that if $A \to B$

is finite étale generically Galois complete normal domain over $A[f^{-1}]$ and tamely ramified in codimension 1, then it's tamely ramified everywhere.

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