MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: KAROL KOZIOL **Email/Phone:** kkoziol@ualberta.ca

Speaker's Name: ROBERT CASS

Talk Title: GEOMETRIC SATAKE

Date: 4 / 8 / 19 **Time:** 3 :30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: THE SPEAKER DISCUSSED THE GEOMETRIC SATAKE EQUIVALENCE AND DEFINED ALL OF ITS RELEVANT CONSTITUENTS. HE ALSO DISCUSSED THE "GLOBAL" VERSION USING THE BD GRASSMANNIAN

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

GEOMETRIC SATAKE — CASS

GR_I CLASSIFIES $(\mathcal{E}, \{x_i\}_{i \in I}, \varphi)$ WHERE

- $\mathcal{E}$ IS A G-BUNDLE
- $\varphi : \mathcal{E} \longrightarrow \mathcal{E}^{TRIV}$ ISOM AWAY FROM Γ_{x_i}

"BEILINSON-DRINFELD GRASSMANNIAN"

- $k = \bar{k}$
- $\Lambda = \bar{\mathbb{Q}}_\ell$
- X/k CURVE
- $v \in |X|$
- G/k (SPLIT) RED'VE
- AFFINE GRASSMANNIAN $GR_v = GR_{\{1\}}\big|_v$

 PARAMETRIZES $\mathcal{E}\big|_{F_v} \xrightarrow{\sim} \mathcal{E}^{TRIV}\big|_{F_v}$, WHERE

 $\mathcal{E}$ IS A G-BUNDLE ON $\mathcal{O}_v$

IDEA PICK $\psi : \mathcal{E}^{TRIV} \longrightarrow \mathcal{E}$. THEN $\varphi \circ \psi \in G(F_v)$, AND DIFF'T CHOICES OF ψ CHANGE $\varphi \circ \psi$ BY RIGHT MULT'N BY $G(\mathcal{O}_v)$

$$\Longrightarrow \quad GR_v(k) = G(F_v)/G(\mathcal{O}_v)$$

EX $G = GL_n$

$$Gr_v(k) = \{ \mathcal{O}_v\text{-LATTICES IN } F_v^{\oplus n} \}$$

IE, $\mathcal{L} \subset F_v^{\oplus n}$ F.G. $\mathcal{O}_v$-SUBMOD

ST $\mathcal{L} \otimes_{\mathcal{O}_v} F_v \cong F_v^{\oplus n}$

$$g \longmapsto g\mathcal{L}_0, \text{ WHERE } \mathcal{L}_0 = \mathcal{O}_v^{\oplus n}$$

FUNCTORS ON k-ALGS

$$LG : R \longmapsto G(R((t)))$$

$$L^+G : R \longmapsto G(R[[t]])$$

THEN AFFINE GRASSMANNIAN $= \underset{\substack{\| \\ Gr}}{LG/L^+G}$ (AS FPQC SHEAVES)

CHOICE OF UNIFORMIZER GIVES

$$Gr_v \cong Gr$$

ALSO HAVE ACTION $L^+G \circlearrowright Gr$

GEOMETRIC SATAKE (LUSZTIG, DRINFELD, GINZBURG, MIRKOVIC-VILONEN, ZHU)

$\exists$ EQUIV OF SYMM'C MONOIDAL CATS

$$\mathrm{Rep}(\hat{G}) \xrightarrow{\text{SAT}} P_{L^+G}(Gr)$$

L^+G-EQUIV PERVERSE SHEAVES

LATER GOAL:

$$\mathrm{SAT}_I : \mathrm{REP}(\hat{G}^I) \xrightarrow{\sim\sim\sim} P_{\mathrm{GOOD}_I}(\mathrm{GR}_I)$$

GEOMETRY OF GR

$$F = k((t)) \supset \mathcal{O} = k[[t]]$$

FIX $T \subset B \subset G$

$$\mu \in X_*(T) \Rightarrow \mu : F^\times \longrightarrow T(F)$$

CARTAN DECOMP:

$$G(F) = \coprod_{\mu \in X_*^+(T)} G(\mathcal{O})\,\mu(t)\,G(\mathcal{O})$$

DOM'T COWTS

$\rightsquigarrow$ $\mathrm{GR}_\mu = L^+G$ ORBIT OF $\mu(t) \subset \mathrm{GR}$

EX $G = GL_2$, $\mu_1 = (1,0) : t \longmapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathrm{GR}_\mu(k) = \{ \mathcal{L} \text{ ST } t\mathcal{L}_0 \subsetneq \mathcal{L} \subsetneq \mathcal{L}_0 \}$$

$$= \{ \text{LINES IN } \mathcal{L}_0 / t\mathcal{L}_0 \}$$

$$\Rightarrow \mathrm{GR}_{\mu_1} \cong \mathbb{P}^1$$

IN GENERAL,

$$\overline{Gr_\mu} = \coprod_{\substack{\lambda \le \mu \\ \lambda \in X_*^+(T)}} Gr_\lambda$$

GET

$$Gr = \varinjlim \overline{Gr_\mu} \qquad + \quad \overline{Gr_\mu} \text{ IS PROJ'IVE}$$

DEFINE $L^n G : R \longmapsto G(R[t]/t^n)$

AND DEFINE

$$P_{L^+G}(\overline{Gr_\mu}) := P(L^n G \backslash \overline{Gr_\mu}) \subset P(\overline{Gr_\mu})$$

$n \gg 0$

AND

$$P_{L^+G}(Gr) := \varinjlim P_{L^+G}(\overline{Gr_\mu})$$

THIS CAT IS SEMISIMPLE, W/ IRREDUCIBLE OBJECTS

$$IC_{\overline{Gr_\mu}}$$

$P_{L^+G}(Gr)$ ALSO HAS A CONVOLUTION PRODUCT

$$Gr \times Gr \xleftarrow{p} LG \times Gr \xrightarrow{q} LG \times^{L^+G} Gr \xrightarrow{m} Gr \times Gr$$

QUOTIENT MAP

QUOTIENT BY L^+G ACTION

$g.(h_1, h_2) = (h_1 g^{-1}, g h_2)$

MULT'N

$(g,h) \longmapsto gh$

p, q ARE L^+G-TORSORS

GIVEN $A, B \in P_{L^+G}(Gr)$, $\exists!\ A \tilde{\boxtimes} B \in P(LG \times^{L^+G} Gr)$ (5)

ST $p^*(A \boxtimes B) \simeq q^*(A \tilde{\boxtimes} B)$

(N.B. ALL FUNCTORS ARE DERIVED)

DEFINE THEN

$$A * B := m_!(A \tilde{\boxtimes} B)$$

EX
$$LG \times^{L^+G} Gr \cong Gr \times Gr$$
$$(g, h) \longmapsto (g, gh)$$

$G = GL_2$, $\mu_1 = (1, 0)$, $\mu_2 = (0, -1)$

THEN $Gr_{\mu_1} \cong \overline{Gr_{\mu_1}}$, AND

$IC_{\overline{Gr_{\mu_i}}} = \overline{\mathbb{Q}}_\ell[1]$ ON SCHEMES $\cong \mathbb{P}^1$

$IC_{\overline{Gr_{\mu_1}}} \tilde{\boxtimes} IC_{\overline{Gr_{\mu_2}}} = \overline{\mathbb{Q}}_\ell[2]$ SUPPORTED ON $Gr_{\mu_1} \tilde{\times} Gr_{\mu_2}$,

WHERE $(Gr_{\mu_1} \tilde{\times} Gr_{\mu_2})(k)$

$$= \{(\mathcal{L}_1, \mathcal{L}_2) : \mathcal{L}_1 \in Gr_{\mu_1}(k),\ \mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq t^{-1}\mathcal{L}_1\}$$

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r] \arrow[d] & Gr_{\mu_1} \tilde{\times} Gr_{\mu_2} \arrow[d, "m"] \\
Gr_{(0,0)} \arrow[r] & \overline{Gr_{(1,-1)}}
\end{tikzcd}$$

RES'N OF SINGULARITIES

$Gr_{(0,0)} = PT$

SO $$m_!\left(\overline{\mathbb{Q}}_\ell[2]\right) = IC_{\overline{GR_{(1,-1)}}} \oplus IC_{GR_{(0,0)}}$$

$$\left(\text{REFLECTS} \quad V_{\mu_1} \otimes V_{\mu_2} = V_{(1,-1)} \oplus V_{(0,0)}\right)$$

//

THIS WAS THE LOCAL STORY. IN ORDER TO PROVE COMMUTATIVITY OF $*$, HAVE TO GLOBALIZE + DEFINE SAT_I

IF $X = \mathbb{A}^1$, $GR_{\{1\}} = GR \times \mathbb{A}^1$, AND CAN DEFINE

$$SAT_{\{1\}} : REP(\hat{G}^{\{1\}}) \longrightarrow P(GR_{\{1\}})$$
$$V \longmapsto SAT(V) \boxtimes \overline{\mathbb{Q}}_\ell[1]$$

IF $I = \{1,2\}$, LET $U = X^2 - \Delta$

$$\begin{tikzcd}
GR_{\{1\}} \arrow[r] \arrow[d] & GR_I \arrow[d] & GR_{\{1\}}^2\big|_U \arrow[l] \arrow[d] \\
X \arrow[r, "\Delta"] & X^2 & U \arrow[l]
\end{tikzcd}$$

<u>EX</u> $G = GL_2$. WANT

$$SAT_I(V_{\mu_1} \boxtimes V_{\mu_2})\big|_U = SAT_{\{1\}}(V_{\mu_1}) \boxtimes SAT_{\{2\}}(V_{\mu_2})\big|_U$$

CLOSURE OF $GR_{\{1\}, V_{\mu_1}} \times GR_{\{2\}, V_{\mu_2}}\big|_U$ RESTRICTS TO

$\overline{GR_{(1,-1)}}$ ON POINTS IN $\Delta(X)$

$$SAT_I(V_{\mu_1} \boxtimes V_{\mu_2})\big|_{\Delta(X)} = SAT(V_{\mu_1} \otimes V_{\mu_2})$$

SET $W = V_{\mu_1} \boxtimes V_{\mu_2}$

$\widetilde{GR}_{I,W}$ PARAMETRIZES

$$\mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}^{TRIV}$$

ISOM AWAY FROM x_2, BDD BY V_{μ_2}

— " — FOR x_1, V_{μ_1}

$\exists$ GLOBAL CONVOLUTION MAP

$$\widetilde{GR}_I \xrightarrow{\text{CONV}} GR_I$$

$$\text{DATA ABOVE} \longmapsto \mathcal{E}_1 \dashrightarrow \mathcal{E}^{TRIV}$$

DEFINE

$$\text{SAT}_I(V_{\mu_1} \boxtimes V_{\mu_2}) = \text{CONV}_!\left(\text{SAT}_{\{1\}}(V_{\mu_1}) \,\tilde{\boxtimes}\, \text{SAT}_{\{2\}}(V_{\mu_2})\right)$$

<u>THM</u> (GAITSGORY) THERE ARE ADDITIVE FUNCTORS

$$\text{REP}(\widehat{G}^I) \longrightarrow \mathcal{P}_{\blacksquare}(GR_I)$$

GLOBAL CONDITION

S.T.

- COMPATIBLE w/ CONVOLUTION
- COMPATIBLE w/ FUSION

MOREOVER

$$\text{REP}(\widehat{G}) \xrightarrow{\text{SAT}_{\{1\}}} \mathcal{P}(GR_{\{1\}}) \xrightarrow{\text{RES}_v} P_{L^+G}(GR)$$

IS COMPAT w/ CLASSICAL SATAKE $(k = \mathbb{F}_q)$

(UP TO TATE TWIST)

FUSION : $J \overset{\varphi}{\twoheadrightarrow} I$ SURJECTION

$$
\begin{tikzcd}
\mathrm{Gr}_I \arrow[r, hook, "\Delta_\varphi"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Gr}_J \arrow[d] \\
X^I \arrow[r, hook] & X^J
\end{tikzcd}
$$

DEFINE

$$\mathrm{Res}_\varphi = \Delta_\varphi^* : \mathrm{Shv}(\mathrm{Gr}_J) \longrightarrow \mathrm{Shv}(\mathrm{Gr}_I)$$

"FUSION"

EX $\{1,2\} \overset{\varphi}{\twoheadrightarrow} \{1\}$

Sat_* INTERTWINES Res_φ w/ $\mathrm{Rep}(\hat{G}^2) \xrightarrow{\Delta_\varphi^*} \mathrm{Rep}(\hat{G})$

EX $\{1,2\} \xrightarrow{\text{SWAP}} \{1,2\}$

$$\mathrm{Rep}(\hat{G}^2) \xrightarrow{\text{SWAP}^*} \mathrm{Rep}(\hat{G}^2)$$

THIS CAN BE USED TO PROVE COMMUTATIVITY