10A

Visualizing Dynamics: Symplectic Maps

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How to make pretty pictures
Allow you to understand the geometry

Concentrate on discrete dynamical system

Standard Map

- Kicked Rotor
  - frictionless, horizontal rotor
  - kick it impulsively, periodic in time
  - easy to solve between kicks & to look at the result of a single kick
  - put these together to get a 1-time step map
  - automatically get a semi-implicit form – needs to be symplectic
- First derived by Taylor & Chirikov
  - model appears in plasma, condensed matter, cyclotron
- this is also a split-step symplectic integrator for a pendulum

Frenkel – Kontorova

- a bunch of particles, connected by springs, in a periodic external potential
  - particles adsorbing onto a periodic surface
- look for equilibria → standard map
- look for a configuration with a fixed spacing between particles

Shows program available on his website – interactive standard map

- $k = 0$ – are horizontal circles ↳ on 2-torus
- k small – circles become wiggly – KAM tori
- k higher – islands form
  - separatrix is where chaos first happens
- $k \sim O(1)$ – large chaotic regions, many islands
- $k \sim O(2)$ – most orbits fill in most of the phase space – still some islands
- $k \sim O(6)$ – only 2 islands left – central island period-doubled
- $k \sim O(10)$ – everything appears chaotic
- $k = 2\pi$ – saddle-center bifurcation → accelerator modes
  - gone by $k = 7.75$

Does this Chaos imply Hyperbolicity?

Arnold's Cat Map

$x' = 2x + y \mod 1$ $\lambda_\pm = \frac{1}{2}(3 \pm \sqrt{5})$

$y' = x + y \mod 1$

uniformly hyperbolic

Is the standard map hyperbolic?

It has many unstable periodic orbits

We can find stable & unstable manifolds for it

Unstable - red, stable - blue - like blood going to/from heart

$k = 0.4$, stable & unstable manifolds almost connect

we can only see slight thickening

$k = 0.7$, start to see crossings - heteroclinic oscillations

$k = 1$, oscillations very visible

quickly fills in chaotic region

$k = 6$, much more wild behavior - quickly runs out of memory

tangle fills much of the phase space

we could also look at manifolds of other periodic orbits

when these intersect, there are homoclinic orbits

resonance zones

take unstable manifold out, then stable back in

possible to escape from a resonance

for $k > k_{cr}$, resonances fill entire space - partition

L no more invariant circles

total area in resonance vs. rotation number is a devil's staircase

if these zones are everywhere, where is the chaos?

Chaotic region is a fat fractal - dim = 2, boundary has non-integer dimension

islands in chaotic region have islands around them - fractal

even for large k, k values with islands are dense

Integrability

not all maps are chaotic

look for an integral that is invariant under the dynamics : $I = I \circ f$

Suris - found / classified integrable 2-dim maps

foliated by invariant curves - contours of I.

Invariant Circles

$k=0$, y = irrational → densely wraps circles

rotation number $\rho = y$

KAM - Diophantine
Twist
Smoothness
} these tori persist for small k

Twist : $\partial x'/\partial y > 0$ velocity increases monotonically with momentum

interates of a twist map aren't twist

Last Invariant Circle

Greene - no invariant circles when $k > k_{cr} = 0.9716\cdots$

last circle's rotation number = golden mean

some remnant persists - invariant Cantor set with gap - Aubrey-Mather Theory

Farey Path

generates rationals - every one between 0 & 1 once

periodic orbits on the tree are irrational numbers

limits of p.o.'s whose periods approach an irrational number should approach an invariant circle with rotational number = that irrational number

chaotic region bound by invariant circle

after k_{cr}, there is transport across the map

Higher Dimensions

open (numerical) tori:

generalization of Farey tree?

notion of robustness / noble numbers

fluid mixing - volume preserving

given a velocity field, how is dye advected?

volume-preserving map

generalization of standard map - 2 angles, 1 action

invariant 2-tori, tubes in various region

version of KAM

open - how do these break up? Aubrey-Mather?

4D Symplectic Map

Froeschlé Map - coupled standard maps

look for some scalar quantity to plot

fast Lyapunov indicator - better conversion than Lyapunov exponent

Slices vs. Projections