

j/w Kathryn Hess: HS '18

Related work: Bohmann, Gerhardt, Hogenhaver, Ziegenhagen (BG-HSZ '18).

Motivation. (to be made precise later.)

Loop spaces.

The free loop space:  $\mathcal{L}X := \text{maps}(S^1, X)$

standard loop space:  $\Omega X := \text{maps}_*(S^1, X)$

Theorem. (Bökstedt-Waldhausen '87) For  $X$  1-connected,

$$\text{THH}(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty \mathcal{L}X.$$

This talk: ~~an improvement~~ / version for coTHH:

Theorem (Kuhn '04, Malkin '17, HS '18)

FCW 1-connected 1-connected weaker condition to be stated.

$$\Sigma_+^\infty \mathcal{L}X \simeq \text{coTHH}(\Sigma_+^\infty X)$$

Why is this an improvement?

- $\Sigma_+^\infty X$  has simpler models than  $\Sigma_+^\infty \Omega X$
- weaker hypotheses on  $X$ .

Connections to algebraic K-theory:

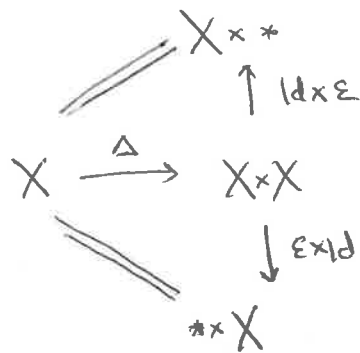
$$\begin{array}{ccc}
 A(X) := K(\Sigma_+^\infty \Omega X) & \xrightarrow{\text{trace}} & \text{THH}(\Sigma_+^\infty \Omega X) \\
 \uparrow & & \uparrow \\
 \text{Waldhausen} & & \text{algebraic k-theory} \\
 \text{K-theory} & & \text{(hard to compute!)} \\
 & & \text{HS '16} \\
 K(\Sigma_+^\infty \Omega X) \simeq K(\Sigma_+^\infty X) & \xrightarrow{\text{trace}} & \text{coTHH}(\Sigma_+^\infty X)
 \end{array}$$

if 1-connected

§ Coalgebras. Primary interest: differential graded setting or spectra. (Shapiro) ②

Coalgebra: comonoid in symmetric monoidal category equipped w/ comital, coassociative multiplication.

E.g.: spaces.  $(\text{Top}, X, *)$ . If  $*$   $\leftarrow X \xrightarrow{\Delta} X \times X$  is a coalgebra structure.

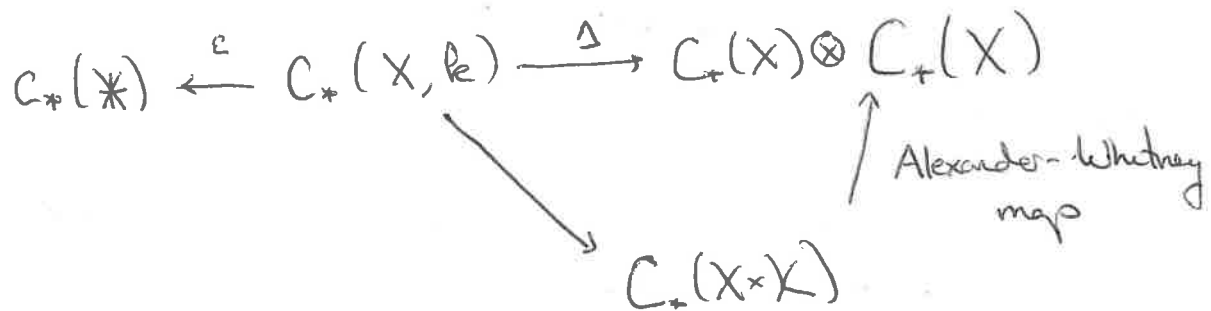


forces the comultiplication to be the diagonal.

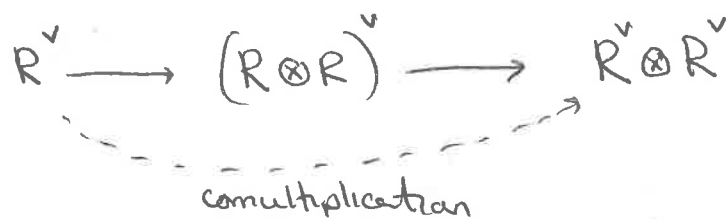
So any space is a coalgebra via the diagonal, and this is the unique coalgebra structure. Note in this case it's cocommutative.

chain complexes.  $(C_n, \otimes, \mathbb{k})$  for  $\mathbb{k}$  a field.

①  $C = C_*(X, \mathbb{k})$ . comultiplication on  $X \rightsquigarrow$  comult. on chains



②  $R$  a finite type dga (differential graded algebra)



(not necessarily comm dga  $\rightsquigarrow$  not necessarily cocomm. coalg.)

(3) (Spectra,  $A, \mathcal{S}$ )  $\times$  a space as in (1).

$$(i) \quad \mathcal{S} = \sum_+^{\infty} \Sigma_+^{\infty} X \xleftarrow{\Sigma_+^{\infty}(\varepsilon)} \Sigma_+^{\infty} X \xrightarrow{\cong} \Sigma_+^{\infty} (X \times X) \cong (\Sigma_+^{\infty} X) \wedge (\Sigma_+^{\infty} X)$$

Based on diagonal, so automatically cocommutative

(ii) [Example works more generally.]

$f: A \rightarrow B$  comm. rings.

$B \wedge_A B$  is a  $B$ -coalgebra as follows:

$$B \wedge_A B \cong B \wedge_A A \wedge_A B \xrightarrow{\cong \wedge 1} B \wedge_A B \wedge_A B$$

$$\Delta \searrow \qquad \qquad \qquad \downarrow \text{SI}$$

$$\qquad \qquad \qquad (B \wedge_A B) \wedge_B (B \wedge_A B)$$

(iii) [Instance of (ii)]

$$\mathcal{S} \rightarrow \text{HFP} \quad \text{HFP} \wedge_{\mathcal{S}} \text{HFP} \quad \text{dual Steenrod algebra.}$$

(iv)  $R$  a compact ring spectrum

$DR = \text{map}(R, \mathcal{S})$  is the Spanier-Whitehead dual.

As for dga's:

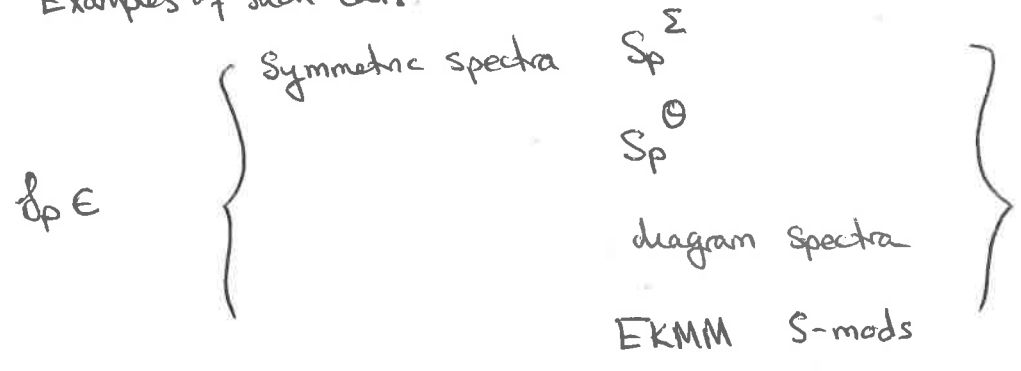
$$DR \rightarrow D(R \star R) \xleftarrow{\cong} DR \wedge DR$$

co-algebra up to homotopy.

(Strict) Symmetric monoidal categories of spectra:

in those that are known, rings and modules work well. Issues with commutativity up to homotopy (for e.g.) don't arise.

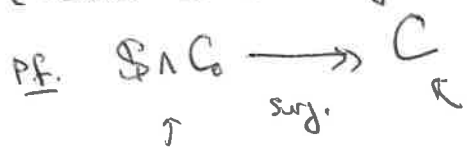
Examples of such cats:



We'll use  $\mathcal{S}p$  for one of the cats.

Homotopically, coalgebra spectra are not understood well in  $\mathcal{S}p$ .

Prop (Peroux-S, 19) Any  $S$ -coalgebra in  $\mathcal{S}p$  is co-commutative



this is cocommutative

therefore this is cocomm.

So coalgebras are not modeled well here. Should move to  $\infty$ -cats or find a new model.

Note: in ongoing work w/ Bowman-Gerberard, developing this. Not for today's talk. Stick to dg setting.

§ Def of  $coTHH$ .

Defn: for  $(C, \Sigma, \Delta)$  a coalgebra, the cotochschild complex

(or cyclic cobar construction) is a cosimplicial object with

$$(coTHH^i(C))^n = C^{\otimes n+1}$$

$$d^i = \Delta \text{ in the } i^{\text{th}} \text{ spot, } i < n+1$$

$$s^i = \varepsilon \text{ in the } i^{\text{th}} \text{ spot}$$

$$d^{n+1} = \int_{C, C^{n+1}} \circ A$$

$$C \xrightarrow[\tau \circ \Delta]{\Delta} C \otimes C \xrightarrow[\tau \circ (\Delta \otimes \Delta)]{\Delta \otimes \Delta} C \otimes C \otimes C \dots$$

(5)

$$\text{coTHH}(C) := \text{holim}_{\leftarrow} (\text{coTHH}^i C)$$

dualization of def of THH. But note: • geometric realization (= localizer)  
has good properties of commutation with  $\otimes$ . This is an added difficulty for  $\text{coTHH}$ .  
• Also, spectral sequences involved don't behave as well.

For  $(C, \varepsilon, \Delta, \eta)$   $\mathbb{1} \xleftarrow{\varepsilon} C$   $\Delta \eta = \eta \otimes \eta$

The cobar complex is  $(\Omega^i C)^n = \mathbb{1} \otimes C^n \otimes \mathbb{1}$

$$d^i = \begin{cases} \eta & \text{in } 0^{\text{th}} \quad i=0 \\ \eta & \text{in } (n+1)^{\text{st}} \quad i=n+1 \\ \Delta & \text{in } i^{\text{th}} \quad \text{otherwise} \end{cases} \quad s^i = \varepsilon \text{ in } i^{\text{th}} \text{ spot}$$

The cobar construction  $\underline{\Omega} C$  is the homotopy inverse limit  $\text{holim}_{\leftarrow} \Omega^i C$

§3 DG  $(\text{Ch}_k, \otimes, k)$  for  $k = \text{field}$ .

Doi '81; Farnan-Solotar '00; Hess-P - Scott '09

$$\text{coHH}_{\oplus}(C) = \text{Tot}_{\oplus} (\text{coTHH}^i C)$$

one can show: that for  $C$  a 1-cann DG,

①  $\text{coHH}_{\oplus}(C) \underset{\text{q-isom}}{\simeq} \text{coTHH}_{\text{Ch}_k}(C)$

② For  $X$  1-cann,  $\underline{\Omega}(C_* X) \underset{\text{quasi-isom}}{\simeq} C_*(\underline{\Omega} X)$

Want to take  $\text{coTHH}$  of ring spectrum, or cat of modules, and say these agree...

Properties of  $\text{coTHH}$

Thm (Hess 16, HPS 09) For  $C$  cann DG,

$$\text{coHH}(C) \underset{\text{q-isom}}{\simeq} \text{HH}(\underline{\Omega} C)$$

This is at the level of homology - can we categorify?

Shpley (C)

Thm (HS '18)

For  $C$  a connected DG coalgebra, there is a Quillen equivalence

$$\text{Comod } C \xrightleftharpoons[\perp]{} \text{Mod } \underline{\Omega} C$$

↑  
homotopy theory developed  
in work of Hess-K

-Riehl-Shpley '17

+ follow-up by Garner-K - Riehl '18

↓  
"all of the  
homotopy theory agrees"

$C \longmapsto$  something weakly equivalent to unit

something w.e. to  
comod.

$\longleftarrow \underline{\Omega} C$

Using this result:

Prop (HS '18). [Agreement for coHH.]  
 $\text{coHH}_*(C) \cong \text{HH}_*(\text{dg cofree } C)$

Pf sketch,  $\text{coHH}(C) \cong \text{HH}(\underline{\Omega} C) \cong \text{HH}(\text{dg free } \underline{\Omega} C)$   
 $\cong \text{HH}(\text{dg cofree } C)$

by Keller  
agreement  
for HH

Note: HH appears because  $\text{dg cofree } C$  is a category.

Another property of HH: Morita invariance. Want analogue for coHH.

Prop (HS '18) If  $C$  and  $D$  are Morita equivalent via a braiding  
(see Berglund-Hess '18), then  $\text{coHH}(C) \cong \text{coHH}(D)$ ,  
↑  
[for def, need notion of dualizability.]

## ⇒ Spectra.

Shipley (7)

For spectra: restrict to suspension spectra to be able to work in strict, rather than  $\infty$ -categorical, setting.

Recall: Eilenberg-Moore SS,  $\Omega X \longrightarrow PX \simeq *$   
 $\downarrow$   
 $X$

The SS converges strongly if  $X$  is connected and  $\pi_* X$  acts nilpotently on  $H_*(\Omega X; \mathbb{Z})$ . We will call such  $X$  EMSS-good.

E.g.: for  $X$  1-connected,  $X$  is EMSS-good.

This is the "weaker condition" stated in theorem stated at the beginning.

Thm. (HS '18, Kuhn '04, Malkiewich '17)

① If  $X$  is EMSS-good, then  $\text{coTHH}(\Sigma_+^\infty X) \simeq \Sigma_+^\infty \mathcal{L}X$ .

② If  $X$  is 1-conn,  $\underline{\Omega}(\Sigma_+^\infty X) \simeq \Sigma_+^\infty \Omega X$

Cor. For 1-conn  $X$ ,  $\text{coTHH}(\Sigma_+^\infty X) \simeq \text{THH}(\Sigma_+^\infty \Omega X)$

Categorification:

Thm (HS '16). For  $X$  connected, there is a Quillen equivalence

$$\text{Comod}_{\Sigma_+^\infty X} \xrightleftharpoons[\perp]{} \text{Mod}_{\Sigma_+^\infty \Omega X}$$

$$\Sigma_+^\infty X \longmapsto \simeq \mathcal{S}$$

$$\simeq \mathcal{S} \longleftarrow \Sigma_+^\infty \Omega X.$$

From this, get agreement:

Cor (HS '18) For  $X$  1-connected,

Shipley (8)

$$\mathrm{coTHH}(\Sigma_+^\infty X) \simeq \mathrm{THH}(\mathrm{Thick}_{\Sigma_+^\infty X}(\mathcal{S}))$$

$$\simeq$$

$$\mathrm{THH}(\Sigma_+^\infty \Omega X) \simeq \mathrm{THH}(\mathrm{Thick}_{\Sigma_+^\infty \Omega X}(\mathcal{S}))$$

B-M '12

Blumberg - Mandell