CoTHH for coalgebras

j/w Kathryn Hess : HS '18

Related work: Bohmann, Gerhardt, Hogenhaven, Ziegenhagen (BGHSZ '18).

Motivation. (to be made precise later.)

Loop spaces.

The free loop space: $\mathcal{L}X := \mathrm{maps}(S^1, X)$

standard loop space: $\Omega X := \mathrm{maps}_*(S^1, X)$

Theorem. (Bökstadt-Waldhausen '87) For X 1-connected,

$$THH(\Sigma^\infty_+ \Omega X) \simeq \Sigma^\infty_+ \mathcal{L}X.$$

This talk: ~~an improvement~~/ version for coTHH:

Theorem (Kuhn '04, Malkiewich '17, HS '18)

FCW 1-connected, 1-connected, weaker condition to be stated.

$$\Sigma^\infty_+ \mathcal{L}X \simeq coTHH(\Sigma^\infty_+ X)$$

Why is this an improvement?

- $\Sigma^\infty_+ X$ has simpler models than $\Sigma^\infty_+ \Omega X$
- weaker hypotheses on X.

Connections to algebraic K-theory:

$$\begin{tikzcd} A(X) := K(\Sigma^\infty_+ \Omega X) \arrow[r, "\text{trace}"] & THH(\Sigma^\infty_+ \Omega X) \arrow[d, equal, "\text{if 1-connected}"] \\ K(\Sigma^\infty_+ \Omega X) \overset{HS '16}{\simeq} K(\Sigma^\infty_+ X) \arrow[r, dashed] & coTHH(\Sigma^\infty_+ X) \end{tikzcd}$$

Waldhausen K-theory

algebraic K-theory (hard to compute!)

§ Coalgebras. Primary interest: differential graded setting or spectra!

Coalgebra: comonoid in symmetric monoidal category equipped w/ counital, coassociative multiplication.

E.g.: • spaces. $(\mathrm{Top}, \underset{\text{product}}{\times}, *)$. If $* \leftarrow X \xrightarrow{\Delta} X \times X$ is a coalgebra structure:

$$\begin{tikzcd} & X \times * \\ X \arrow[ur, equal] \arrow[r, "\Delta"] \arrow[dr, equal] & X \times X \arrow[u, "\mathrm{Id} \times \varepsilon"'] \arrow[d, "\varepsilon \times \mathrm{Id}"] \\ & * \times X \end{tikzcd}$$

forces the comultiplication to be the diagonal.

So any space is a coalgebra via the diagonal, and this is the unique coalgebra structure. Note in this case it's cocommutative.

• chain complexes. $(\mathrm{Ch}_k, \otimes, k)$ for k a field.

① $C = C_*(X, k)$. comultiplication on $X \rightsquigarrow$ comult. on chains

$$\begin{tikzcd} C_*(*) & C_*(X, k) \arrow[l, "\varepsilon"'] \arrow[r, "\Delta"] \arrow[dr] & C_*(X) \otimes C_*(X) \\ & & C_*(X \times X) \arrow[u, "\text{Alexander-Whitney map}"'] \end{tikzcd}$$

② R a finite type dga (differential graded algebra)

$$\begin{tikzcd} R^\vee \arrow[r] \arrow[rr, bend right, dashed, "\text{comultiplication}"'] & (R \otimes R)^\vee \arrow[r] & R^\vee \otimes R^\vee \end{tikzcd}$$

(not necessarily comm dga $\rightsquigarrow$ not necessarily cocomm. coalg.)

Examples continued.

③ (Spectra, A, $\mathbb{S}$) X a space as in ①.

(i)
$$\mathbb{S} = \Sigma^\infty_+ * \xleftarrow{\Sigma^\infty_+(\varepsilon)} \underset{\displaystyle \Sigma^\infty(X_+)}{\overset{\|}{\Sigma^\infty_+ X}} \longrightarrow \Sigma^\infty_+(X \times X) \simeq (\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ X)$$

Based on diagonal, so automatically cocommutative

(ii) [Example works more generally.]

$f: A \to B$ comm. rings.

$B \wedge_A B$ is a B-coalgebra as follows:

$$\begin{tikzcd} B \wedge_A B \simeq B \wedge_A A \wedge_A B \arrow[r, "1 \wedge f \wedge 1"] \arrow[dr, "\Delta"'] & B \wedge_A B \wedge_A B \arrow[d, phantom, "\wr"] \\ & (B \wedge_A B) \wedge_B (B \wedge_A B) \end{tikzcd}$$

(iii) [Instance of (ii)]

$\mathbb{S} \longrightarrow H\mathbb{F}_p$ $\quad H\mathbb{F}_p \wedge_{\mathbb{S}} H\mathbb{F}_p$ dual Steenrod algebra.

(iv) R a compact ring spectrum

$DR = map(R, \mathbb{S})$ is the Spanier-Whitehead dual.

As for dga's:

$$\begin{tikzcd} DR \arrow[r] \arrow[rr, bend right, dashed] & D(R \times R) & DR \wedge DR \arrow[l, "\simeq"'] \end{tikzcd}$$

co-algebra up to homotopy.

(Strict) Symmetric monoidal categories of spectra:

In those that are known, rings and modules work well.

Issues with commutativity up to homotopy (for e.g.) don't arise.

Examples of such cats:

$$\mathcal{S}p \in \left\{ \begin{array}{ll} \text{Symmetric spectra} & Sp^{\Sigma} \\ & Sp^{\Theta} \\ & \text{diagram spectra} \\ \text{EKMM} & S\text{-mods} \\ & \vdots \end{array} \right\}$$

We'll use $\mathcal{S}p$ for one of the cats.

Homotopically, coalgebra spectra are not understood well in $\mathcal{S}p$.

Prop (Peroux-S, 19) Any $\mathbb{S}$-coalgebra in $\mathcal{S}p$ is co-commutative

Pf. $\mathbb{S} \wedge C_0 \xrightarrow[\text{surj.}]{} C$

this is cocommutative — therefore this is cocomm.

So coalgebras are not modeled well here. Should move to ∞-cats or find a new model.

Note: in ongoing work w/ Bowman-Gerhardt, developing this. Not for today's talk. Stick to dg setting.

§ Def of coTHH.

Defn: For (C, ε, Δ) a coalgebra, the coHochschild complex (or cyclic cobar construction) is a cosimplicial object with

$$(\text{coTHH}^{\bullet}(C))^n = C^{\otimes n+1}$$

$d^i = \Delta$ in the i^{th} spot, $i < n+1$

$s^i = \varepsilon$ in the i^{th} spot

$d^{n+1} = \tau_{C, C^{n+1}} \circ \Delta$

$$C \underset{T\circ\Delta}{\overset{\Delta}{\rightrightarrows}} C\otimes C \begin{array}{c}\xrightarrow{\Delta\otimes 1}\\ \xrightarrow{1\otimes\Delta}\\ \xrightarrow{T\otimes(\Delta\otimes 1)}\end{array} C\otimes C\otimes C \cdots$$

$$\mathrm{coTHH}(C) := \underset{\leftarrow}{\mathrm{holim}}\,(\mathrm{coTHH}^{\bullet} C)$$

dualization of def of THH. But note: • geometric realization (= hocolim) has good properties of commutation with $\otimes$. This is an added difficulty for coTHH

• Also, spectral sequences involved don't behave as well.

For $(C, \varepsilon, \Delta, \eta)$ $\quad \mathbb{I} \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} C \quad \Delta\eta = \eta\otimes\eta$

The cobar complex is $(\Omega^{\bullet} C)^n = \mathbb{I}\otimes C^n \otimes \mathbb{I}$

$d^i = \eta$ in 0^{th} $i=0$ $\qquad s^i = \varepsilon$ in ith spot

η in $n+1^{st}$ $i = n+1$

Δ in ith otherwise

The cobar construction ΩC is the homotopy inverse limit $\underset{\leftarrow}{\mathrm{holim}}\, \Omega^{\bullet} C$

§3 DG. $\quad (\mathrm{Ch}_k, \otimes, k)$ for k = field.

Doi '81; Farinati - Solotar '00; Hess - P - Scott '09.

$$\mathrm{coHH}_{\oplus}(C) = \mathrm{Tot}_{\oplus}(\mathrm{coTHH}^{\bullet} C)$$

one can show: that for C a 1-conn DG,

① $\mathrm{coHH}_{\oplus}(C) \underset{q\text{-isom}}{\simeq} \mathrm{coTHH}_{\mathrm{Ch}_k}(C)$

② For X 1-conn, $\Omega(C_* X) \underset{\text{quasi-isom}}{\simeq} C_*(\Omega X)$

Want to take coTHH of ring spectrum, or cat of modules, and say these agree...

Properties of coTHH.

Thm (Hess 16, HPS 09) For C conn DG,

$$\mathrm{coHH}(C) \underset{q\text{-isom}}{\simeq} \mathrm{HH}(\Omega C).$$

This is at the level of homology — can we categorify? Shipley ⑥

<u>Thm</u> (HS'18)
For C a connected DG coalgebra, there is a Quillen <u>equivalence</u>

$$\mathrm{Comod}_C \underset{\perp}{\rightleftarrows} \mathrm{Mod}_{\Omega C}$$

" all of the homotopy theory agrees "

homotopy theory developed in work of Hess-K-Riehl-Shipley '17 + follow-up by Garner-K-Riehl '18.

$C \longmapsto$ something weakly equivalent to unit

something w.e. to counit. $\longleftarrow$ ΩC

<u>Using this result:</u>

<u>Prop</u> (HS '18). [Agreement for coHH.]

$$\mathrm{coHH}_*(C) \simeq \mathrm{HH}_*(\mathrm{dg\ cofree}_C)$$

<u>Pf sketch</u>. $\mathrm{coHH}(C) \simeq \mathrm{HH}(\Omega C) \simeq \mathrm{HH}(\mathrm{dg\ free}_{\Omega C})$ by Keller agreement for HH

$$\simeq \mathrm{HH}(\mathrm{dg\ cofree}_C)$$

Note: HH appears because dg cofree_C is a <u>category</u>.

Another property of HH: <u>Morita invariance</u>. Want analogue for coHH.

<u>Prop</u> (HS'18) If C and D are Morita equivalent via a braiding (see Berglund-Hess '18), then $\mathrm{coHH}(C) \simeq \mathrm{coHH}(D)$.

[For def, need notion of dualizeability.]

§ Spectra.

For spectra: restrict to suspension spectra to be able to work in strict, rather than ∞-categorical, setting.

Recall: Eilenberg-Moore SS,

$$\begin{array}{ccc} \Omega X & \longrightarrow & PX \simeq * \\ & & \downarrow \\ & & X \end{array}$$

The SS converges strongly if X is connected and $\pi_1 X$ acts nilpotently on $H_*(\Omega X;\mathbb{Z})$. We will call such X EMSS-good.

E.g.: for X 1-connected, X is EMSS-good.

This is the "weaker condition" stated in theorem stated at the beginning.

Thm. (HS '18, Kuhn '04, Malkiewich '17)

(1) If X is EMSS-good, then $\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ \mathcal{L}X$.

(2) If X is 1-conn, $\underline{\Omega}(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ \Omega X$

Cor. For 1-conn X, $\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \mathrm{THH}(\Sigma^\infty_+ \Omega X)$

Categorification:

Thm (HS '16). For X connected, there is a Quillen equivalence

$$\mathrm{Comod}_{\Sigma^\infty_+ X} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{Mod}_{\Sigma^\infty_+ \Omega X}$$

$$\Sigma^\infty_+ X \longmapsto \simeq \mathbb{S}$$

$$\simeq \mathbb{S} \longleftarrow\!\shortmid \Sigma^\infty_+ \Omega X.$$

From this, get agreement:

<u>Cor</u> (HS '18) For X 1-connected,

$$\begin{tikzcd}
\mathrm{coTHH}(\Sigma^\infty_+ X) \arrow[dr, dash, "\simeq"] & \simeq & \mathrm{THH}\big(\mathrm{Thick}_{\Sigma^\infty_+ X}(\mathbb{S})\big) \arrow[dr, dash, "\simeq\ \text{HS '18}"] & \\
& \mathrm{THH}(\Sigma^\infty_+ \Omega X) & \underset{\text{B-M '12}}{\simeq} & \mathrm{THH}(\mathrm{Thick}_{\Sigma^\infty_+ \Omega X}\, \Sigma^\infty_+ \Omega X)
\end{tikzcd}$$

Blumberg – Mandell